KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Unit P1 :
Algebra and Functions 1

(part 1)

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

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(part 2)

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**italics indicate summer task work*

Prior knowledge check

Mixed exercise 1

1 Simplify:

a $y^3 \times y^5$ **b** $3x^2 \times 2x^5$ **c** $(4x^2)^3 \div 2x^5$ **d** $4b^2 \times 3b^3 \times b^4$

2 Expand and simplify if possible:

a $(x + 3)(x - 5)$ **b** $(2x - 7)(3x + 1)$ **c** $(2x + 5)(3x - y + 2)$

3 Expand and simplify if possible:

a $x(x + 4)(x - 1)$ **b** $(x + 2)(x - 3)(x + 7)$ **c** $(2x + 3)(x - 2)(3x - 1)$

4 Expand the brackets:

a $3(5y + 4)$ **b** $5x^2(3 - 5x + 2x^2)$ **c** $5x(2x + 3) - 2x(1 - 3x)$ **d** $3x^2(1 + 3x) - 2x(3x - 2)$

5 Factorise these expressions completely:

a $3x^2 + 4x$ **b** $4y^2 + 10y$ **c** $x^2 + xy + xy^2$ **d** $8xy^2 + 10x^2y$

6 Factorise:

a $x^2 + 3x + 2$ **b** $3x^2 + 6x$ **c** $x^2 - 2x - 35$ **d** $2x^2 - x - 3$
e $5x^2 - 13x - 6$ **f** $6 - 5x - x^2$

7 Factorise:

a $2x^3 + 6x$ **b** $x^3 - 36x$ **c** $2x^3 + 7x^2 - 15x$

8 Simplify:

a $9x^3 \div 3x^{-3}$ **b** $(4^{\frac{3}{2}})^{\frac{1}{3}}$ **c** $3x^{-2} \times 2x^4$ **d** $3x^{\frac{1}{3}} \div 6x^{\frac{2}{3}}$

9 Evaluate:

a $\left(\frac{8}{27}\right)^{\frac{2}{3}}$ **b** $\left(\frac{225}{289}\right)^{\frac{3}{2}}$

10 Simplify:

a $\frac{3}{\sqrt{63}}$ **b** $\sqrt{20} + 2\sqrt{45} - \sqrt{80}$

11 **a** Find the value of $35x^2 + 2x - 48$ when $x = 25$.

b By factorising the expression, show that your answer to part **a** can be written as the product of two prime factors.

12 Expand and simplify if possible:

a $\sqrt{2}(3 + \sqrt{5})$ **b** $(2 - \sqrt{5})(5 + \sqrt{3})$ **c** $(6 - \sqrt{2})(4 - \sqrt{7})$

13 Rationalise the denominator and simplify:

a $\frac{1}{\sqrt{3}}$ **b** $\frac{1}{\sqrt{2} - 1}$ **c** $\frac{3}{\sqrt{3} - 2}$ **d** $\frac{\sqrt{23} - \sqrt{37}}{\sqrt{23} + \sqrt{37}}$ **e** $\frac{1}{(2 + \sqrt{3})^2}$ **f** $\frac{1}{(4 - \sqrt{7})^2}$

14 **a** Given that $x^3 - x^2 - 17x - 15 = (x + 3)(x^2 + bx + c)$, where b and c are constants, work out the values of b and c.

b Hence, fully factorise $x^3 - x^2 - 17x - 15$.

E **15** Given that $y = \frac{1}{64}x^3$ express each of the following in the form kx^n, where k and n are constants.

a $y^{\frac{1}{3}}$ **(1 mark)**

b $4y^{-1}$ **(1 mark)**

E/P **16** Show that $\frac{5}{\sqrt{75} - \sqrt{50}}$ can be written in the form $\sqrt{a} + \sqrt{b}$, where a and b are integers. **(5 marks)**

E **17** Expand and simplify $(\sqrt{11} - 5)(5 - \sqrt{11})$. **(2 marks)**

E **18** Factorise completely $x - 64x^3$. **(3 marks)**

E/P **19** Express 27^{2x+1} in the form 3^y, stating y in terms of x. **(2 marks)**

E/P **20** Solve the equation $8 + x\sqrt{12} = \frac{8x}{\sqrt{3}}$

Give your answer in the form $a\sqrt{b}$ where a and b are integers. **(4 marks)**

P **21** A rectangle has a length of $(1 + \sqrt{3})$ cm and area of $\sqrt{12}$ cm^2.
Calculate the width of the rectangle in cm.
Express your answer in the form $a + b\sqrt{3}$, where a and b are integers to be found.

E **22** Show that $\frac{(2 - \sqrt{x})^2}{\sqrt{x}}$ can be written as $4x^{-\frac{1}{2}} - 4 + x^{\frac{1}{2}}$. **(2 marks)**

E/P **23** Given that $243\sqrt{3} = 3^a$, find the value of a. **(3 marks)**

E/P **24** Given that $\frac{4x^3 + x^{\frac{5}{2}}}{\sqrt{x}}$ can be written in the form $4x^a + x^b$, write down the value of a and the value of b. **(2 marks)**

Challenge

a Simplify $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$.

b Hence show that $\frac{1}{\sqrt{1} + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{4}} + \ldots + \frac{1}{\sqrt{24} + \sqrt{25}} = 4$

Prior knowledge check

Problem solving Set A

Bronze

a Simplify $\sqrt{147} - \sqrt{75}$ giving your answer in the form $a\sqrt{3}$, where a is an integer. **(2 marks)**

b Hence, or otherwise, simplify $\frac{24\sqrt{2}}{\sqrt{147} - \sqrt{75}}$ giving your answer in the form $b\sqrt{6}$, where b is an integer. **(2 marks)**

Silver

a Expand and simplify $(11 + \sqrt{5})(\sqrt{5} + 1)$ giving your answer in the form $a + b\sqrt{5}$, where a and b are integers. **(2 marks)**

b Hence, or otherwise, simplify $\frac{11 + \sqrt{5}}{\sqrt{5} - 1}$ giving your answer in the form $c + d\sqrt{5}$, where c and d are integers. **(2 marks)**

Gold

Simplify $\frac{6\sqrt{3} - 4}{5 - \sqrt{3}}$, giving your answer in the form $p\sqrt{3} - q$, where p and q are positive rational numbers. **(4 marks)**

Problem solving Set B

Bronze

a Express 4^{x+2} in the form 2^y, stating y in terms of x. **(2 marks)**

b Hence, or otherwise, solve the equation $4^{x+2} = 32$. **(2 marks)**

Silver

a Rewrite the equation $(4^{x-1})^2 - 6(4^{x-1}) + 8 = 0$ in the form $y^2 + by + c = 0$, where $y = 4^{x-1}$ and b and c are constants to be found. **(1 mark)**

b Factorise your equation from part **a** and hence, or otherwise, solve for x. You must show each step of your working. **(3 marks)**

Gold

Solve the following equation, showing each step of your working:

$(9^{x-1})^2 - 30(9^{x-1}) + 81 = 0$ **(5 marks)**

Now try this → Exam question bank Q1, Q2, Q4, Q24, Q36, Q46, Q64

AS 2020

3. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

(i) Solve the equation

$$x\sqrt{2} - \sqrt{18} = x$$

writing the answer as a surd in simplest form.

(3)

(ii) Solve the equation

$$4^{3x-2} = \frac{1}{2\sqrt{2}}$$

(3)

AS 2021

2. **In this question you should show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

Given

$$\frac{9^{x-1}}{3^{y+2}} = 81$$

express y in terms of x, writing your answer in simplest form.

(3)

Prior knowledge check

2.2 Completing the square

1 Complete the square for these expressions:

a x^2+2x b x^2-8x

c x^2-12x

Hint $x^2+bx=\left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2$

2 Complete the square for these expressions:

a $x^2-8x+12$ b x^2-x-12

c x^2-3x-4

Hint When a quadratic expression is in completed square form, x only appears once. The constants in your answer do not have to be integers.

3 Write each of these expressions in the form $a(x+b)^2+c$, where a, b and c are constants to be found:

Hint Start by factorising the first two terms of each expression.

a $3x^2-12x+17$ b $5x^2-10x+12$ c $3x^2-7x+2$

4 Solve these quadratic equations by completing the square. Leave your answers in surd form.

a $x^2-2x-10=0$ b $5x^2+12x+6=0$

c $3x^2-7x-2=0$

Hint Write the left-hand side of each equation in completed square form. Then use inverse operations to make x the subject of the equation. Remember to include $\pm$ when you take square roots of both sides of the equation.

E/P 5 $\mathrm{f}(x)=x^2-7x-2,\ x\in\mathbb{R}$

a Express $\mathrm{f}(x)$ in the form $(x-a)^2-b$, where a and b are constants. **(2 marks)**

b Hence write down the minimum value of $\mathrm{f}(x)$. **(1 mark)**

E 6 $\mathrm{h}(x)=4-2x-3x^2,\ x\in\mathbb{R}$

a Express $\mathrm{h}(x)$ in the form $a(x+b)^2+c$, where a, b and c are constants. **(2 marks)**

b Hence, or otherwise, find the exact solutions to $\mathrm{h}(x)=0$. **(2 marks)**

E 7 $6x-2-x^2\equiv q-(x+p)^2$, where p and q are integers.

a Find the value of p and the value of q. **(3 marks)**

b Hence, or otherwise, solve the equation $6x-2-x^2=0$. **(2 marks)**

7.1 Algebraic fractions

1 Simplify these fractions:

a $\dfrac{6x^4+8x^3-4x^2}{2x}$ b $\dfrac{12x-24x^2+9x^3}{3x}$

c $\dfrac{8x^6-4x^5+2x^4}{-2x^2}$

Hint Divide each term in the numerator by the denominator.

2 Write each of these fractions in their simplest form:

a $\dfrac{x^2-9}{x-3}$ b $\dfrac{x^2-3x+2}{x-2}$

c $\dfrac{2x^2+5x-3}{x+3}$

Hint To simplify an algebraic fraction, factorise the numerator and denominator where possible and cancel common factors.

3 Write each of these fractions in their simplest form:

a $\dfrac{2x^2-3x-2}{x^2+x-6}$ b $\dfrac{3x^2+x-2}{2x^2+5x+3}$ c $\dfrac{6x^2+11x-10}{2x^2-3x-20}$

E/P 4 Show that $\dfrac{15x^2-16x-15}{3x^2-5x}$ can be written in the form $A+\dfrac{B}{x}$, where A and B are integers to be found. **(3 marks)**

E/P 5 $\dfrac{x^2-2x+k}{x^2+2x-3}=\dfrac{x-5}{x-1}$ where k is a constant. Work out the value of k. **(3 marks)**

E/P 6 $\dfrac{4x^2+8x-5}{6x^2-7x+k}=\dfrac{2x+5}{3x-2}$ where k is a constant. Work out the value of k. **(4 marks)**

E/P 7 $\dfrac{6x^3-2x^2-4x}{2x^2-7x+5}=\dfrac{ax(bx+c)}{px+q}$, where a, b, c, p and q are constants.

Work out the values of a, b, c, p and q. **(5 marks)**

Prior knowledge check

2.1 Solving quadratic equations

1 Solve the following equations using factorisation:

a $x^2 + 7x + 10 = 0$ **b** $x^2 - 5x - 24 = 0$

c $x^2 + 6x = 0$

Hint The factorised quadratic equation $(x - a)(x - b) = 0$ will have solutions $x = a$ and $x = b$.

2 Solve the following equations using factorisation:

a $7x^2 = 21x$ **b** $4x^2 = 49$

c $4x^2 - 20x + 25 = 0$ **d** $2x^2 - 5x + 2 = 0$

Hint Write the equation in the form $ax^2 + bx + c = 0$ before factorising. Solutions to quadratic equations do not have to be integers.

3 Solve the following equations:

a $(x - 4)^2 = 0$ **b** $x(2x - 1) = 21$

c $4x^2 + 4x + 24 = 2x^2 - 10x$

Hint A quadratic equation may have only one solution, called a repeated root: If $(x - a)^2 = 0$, then $x = a$.

4 Solve the following equations using the quadratic formula.

Give your answers exactly, leaving them in surd form.

a $x^2 + 8x + 6 = 0$ **b** $x^2 + 4x = 1$ **c** $2x^2 - 12x + 15 = 0$

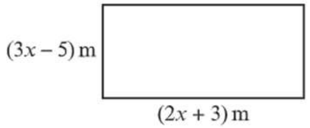
Hint The solutions to $ax^2 + bx + c = 0$ are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

5 Solve the following equations using a suitable method.
Where necessary, give your answers to 3 significant figures:

a $10x - x^2 - 9 = 0$ **b** $64x^2 = 100$ **c** $3x^2 + 10x - 25 = 0$

d $12x = 3x^2 + 5$ **e** $x = \sqrt{5x}$ **f** $2x^2 + 6x + 1 = 0$

(E) 6 The length of a rectangular carpet is $(2x + 3)$ metres and its width is $(3x - 5)$ metres.



The carpet has an area of 20 m^2.

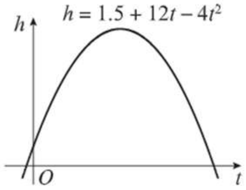
a Show that $6x^2 - x - 35 = 0$. **(2 marks)**

b Hence find the length and width of the carpet, in metres. **(2 marks)**

(E) 7 Solve the equation $x^2 + 4x + 1 = 0$. Write your answer in the form $a \pm \sqrt{b}$, where a and b are integers to be found. **(2 marks)**

(E) 8 Solve the equation $0.1x^2 + 1.6x = 0.8$. Give your answers to 3 significant figures. **(2 marks)**

(E/P) 9 The height, h metres, of a ball at time t seconds after it is thrown up in the air can be modelled by the equation $h = 1.5 + 12t - 4t^2$.



a Find how many seconds it takes for the ball to hit the ground again. Give your answer correct to 3 significant figures. **(2 marks)**

b Find the times when the height of the ball is 9.5 m. **(2 marks)**

1.5 Solving Quadratic Equations (including hidden quadratics)

A hidden quadratic expression is one of the form

$$a[\,f(x)]^2 + b[f(x)\,] + c$$

where $f(x)$ is some function of x and a, b and c are constants

$$x^4 - 13x^2 + 36$$

can be written as

$$(\,x^2)^2 - 13(\,x^2) + 36$$

So $f(x) = x^2$

We can use index laws to rewrite x^4 as $(x^2)^2$

$$x - 13\sqrt{x} + 36$$

can be written as

$$(\sqrt{x})^2 - 13(\sqrt{x}\,) + 36$$

So $f(x) = \sqrt{x}$

Recall that $(\sqrt{x})^2 = \sqrt{x}\times\sqrt{x} = x$

Notes

Factorise $x^4 - 13x^2 + 36$

Using the substitution $y = x^2$
we can rewrite the hidden quadratic as:

$$y^2 - 13y + 36$$

Recall, we need two factors of $+36$ that sum to -13

We can now factorise:

$$(y-9)(y-4)$$

These are -9 and -4

Finally, we can write the factorised expression in terms of x:

$$x^4 - 13x^2 + 36 = \boldsymbol{(x^2 - 9)(x^2 - 4)}$$

Factorise $4\times 3^{2x} - 11\times 3^x - 3$

Using the substitution $y = 3^x$
we can rewrite the hidden quadratic as:

$$4y^2 - 11y - 3$$

We can now factorise by splitting the middle term:

$$4y^2 - 12y + y - 3$$
$$= 4y(y-3) + 1(y-3)$$
$$= (4y+1)(y-3)$$

Since $a\times c = 4\times -3 = -12$, we need two factors of -12 that sum to -11

These are -12 and $+1$

Finally, we can write the factorised expression in terms of x:

$$4\times 3^{2x} - 11\times 3^x - 3 = \boldsymbol{(4\times 3^x + 1)(3^x - 3)}$$

Worked Example

Solve the equations

a) $2x - 3\sqrt{x} = 35$

b) $4y = 12\sqrt{x} + 27$

Dr Frost 485b

Worked Example

Solve the equations

a) $2 \times 2^{2x+1} = -15 \times 2^x + 4$

b) $8 \times 4^x - 9 \times 2^x + 1 = 0$

Fill in the Gaps

Hidden Quadratic	Substitution $y = \cdots$	Quadratic in terms of y	Factorise and Solve Quadratic	Solutions to Hidden Quadratic
$x^4 - 6x^2 + 8 = 0$	$y = x^2$	$y^2 - 6y + 8 = 0$	$(y-4)(y-2) = 0$ $y = 4, y = 2$	$x^2 = 4, x^2 = 2$ $x = \pm 2, x = \pm\sqrt{2}$
$a^6 - 28a^3 + 27 = 0$	$y = a^3$			
$b + \sqrt{b} - 12 = 0$				
$2^{2x} - 5 \times 2^x + 4 = 0$				
$4w^4 - 13w^2 + 9 = 0$				
$9 \times 3^{2z} - 82 \times 3^z + 9 = 0$				
$6t^{2/3} - 5t^{1/3} - 4 = 0$				

Fill in the Gaps

Original Equation in x	Substitution	Quadratic in y	Solutions for y	Solutions for x
$x^4 - 10x^2 + 21 = 0$	$y = x^2$	$y^2 - 10y + 21 = 0$	$y = 7, y = 3$	$x = \pm\sqrt{7}, \pm\sqrt{3}$
$x^6 = 7x^3 + 8$	$y = x^3$		$y = 8, y = -1$	
$x - 3\sqrt{x} - 10 = 0$				$x = 25, x = 4$
$2^{2x} - 6 \times 2^x + 8 = 0$	$y = 2^x$			
$\sqrt{x} + \frac{1}{\sqrt{x}} = 2$				
$9^x - 28 \times 3^x + 27 = 0$				
$x\sqrt[3]{x} - 13x^{\frac{2}{3}} + 36 = 0$	$y = x^{\frac{2}{3}}$			
$x^3 + 9x + \frac{20}{x} = 0$				
$\left(x - \frac{6}{x}\right)^2 - 6\left(x - \frac{6}{x}\right) + 5 = 0$	$y = \left(x - \frac{6}{x}\right)$			

Fluency Practice

1 Identify a suitable substitution of the form $y = f(x)$ for each of these hidden quadratic expressions:

a $x^6 - 6x^3 + 8$?

b $x + 5\sqrt{x} + 4$?

c $2x^{10} - 9x^5 + 10$?

d $(3^x)^2 + 8(3^x) + 7$?

e $4\times 2^{2x} - 33\times 2^x + 8$?

f $2(x-3)^2 + 5(x-3) + 3$?

2 Factorise:

a $z + \sqrt{z} - 12$?

b $a^4 - 9a^2 + 20$?

c $3^{2x} - 4\times 3^x - 21$?

d $2d^6 + 11d^3 + 15$?

e $5(3x-7)^2 - 3(3x-7) - 2$?

f $10\times 4^{2w} - 7\times 4^w + 1$?

3 Factorise fully:

a $a^4 - 9$?

b $16 - w^6$?

c $7^{2x} - 121$?

d $4y^8 - 49$?

e $z^4 - 81$?

4 Solve:

a $w^4 - 17w^2 + 16 = 0$?

b $x - 9\sqrt{x} + 20 = 0$?

c $z^6 - 28z^3 + 27 = 0$?

d $4d^4 - 17d^2 + 4 = 0$?

e $6t + 5\sqrt{t} - 4 = 0$?

Fluency Practice

5 *[OCR C1 June 2006 Q6i]*
Solve the equation $x^4 - 10x^2 + 25 = 0$

?

6 *[OCR C1 June 2010 Q5]*
Find the real roots of the equation
$4x^4 + 3x^2 - 1 = 0$

?

7 Solve:

a $2^{2x} - 5\times2^x + 4 = 0$?

b $3^{2y} - 4\times3^y + 3 = 0$?

c $16^w - 17\times4^w + 16 = 0$?

d $9\times9^x - 28\times3^x + 3 = 0$?

8 *[OCR A2 June 2018 P1 Q3]*
Find the two real roots of the equation
$x^4 - 5 = 4x^2$ Give the roots in an exact form.

?

9 *[OCR C1 Jan 2008 Q10ii]*
Given that $f(x) = 8x^3 + \frac{1}{x^3}$, solve the equation $f(x) = -9$

?

10 *[WJEC AS Level June 2018 Unit 1 Q17c]*
Express $4^x - 10\times2^x$ in terms of y, where $y = 2^x$. Hence solve the equation $4^x - 10\times2^x = -16$

?

☠ Solve:

a $3z^{2/3} + z^{1/3} - 2 = 0$?

b $2^{2w+1} - 17\times2^w + 8 = 0$

?

Fluency Practice

To spot a quadratic in disguise you are looking for an equation where the power on one of the variables is twice that on the other. For example

$$(\text{whatever})^{18} + 4(\text{whatever})^9 - 5.$$

This can then factorise to

$$((\text{whatever})^9 + 5)((\text{whatever})^9 - 1),$$

or you can complete the square to

$$((\text{whatever})^9 + 2)^2 - 9.$$

Solve the following:

1. $x^4 - 13x^2 + 36 = 0.$
2. $x^4 - 15x^2 - 16 = 0.$
3. $x^4 + 5x^2 + 6 = 0.$
4. $x^6 + 7x^3 = 8.$
5. $2x^4 = x^2 + 1.$
6. $x + 3 = 4\sqrt{x}.$
7. $12x^4 = 2x^2 + 4.$
8. $2(\sin x)^2 + \sin x - 1 = 0$ in range $0 < x < 360.$
9. $\frac{4x^4+144}{73} = x^2.$
10. $2(\cos x)^2 + (\cos x) = 6.$
11. $x = 2\sqrt{x} + 3.$
12. $6x^{2/3} + 5x^{1/3} - 4 = 0.$
13. $(x^2 - 4x + 1)^2 + (x^2 - 4x + 1) - 12 = 0.$
14. $2\theta + 15 = 11\sqrt{\theta}.$
15. $x^2 + \frac{72}{x^2} = 17.$
16. $\sqrt{z} - 2\sqrt[4]{z} = 3.$
17. $2^{2x} - 12 \times 2^x + 32 = 0.$
18. $t = 4\sqrt{t} + 1.$
19. $2^{2x} + 8 = 9 \times 2^x.$
20. $2\sqrt{x} + \frac{9}{\sqrt{x}} = 9.$
21. $\left(\frac{1}{x}\right)^2 + 1 = 8\left(\frac{1}{x}\right).$ [Do this question in two ways.]
22. $2(\cos\theta)^2 = 8\cos\theta + 21.$
23. $x^8 + 4x^4 = 3.$
24. $2^{2x} + 1 = 2^{x+1}.$
25. $2^{2x} + 128 = 3 \times 2^{x+3}.$
26. $81 + 3^{2x+1} = 4 \times 3^{x+2}.$
27. $a^{2x} + a^4 = a^{x+1} + a^{x+3}.$

Only attempt the following if you have studied logarithms.

28. $2^{2x} - 13 \times 2^x + 42 = 0.$
29. $4^{2x} - 9 \times 4^x + 14 = 0.$
30. $3^{2x} + 10 = 7 \times 3^x.$
31. $2^{2x} - 5 \times 2^{x+1} + 25 = 0.$
32. $3^{2x} - 3^{x+2} + 20 = 0.$

T. 27 2E Qs 7,8 P.9 2.3 Qs 6,7

Exam Q AS 2019

2. Find, using algebra, all real solutions to the equation

(i) $16a^2 = 2\sqrt{a}$

(4)

(ii) $b^4 + 7b^2 - 18 = 0$

(4)

Your Turn

Find, using algebra, all real solutions to the equation

$$a^4 - 2a^2 - 80 = 0$$

(4)

1.6 Quadratic Graphs

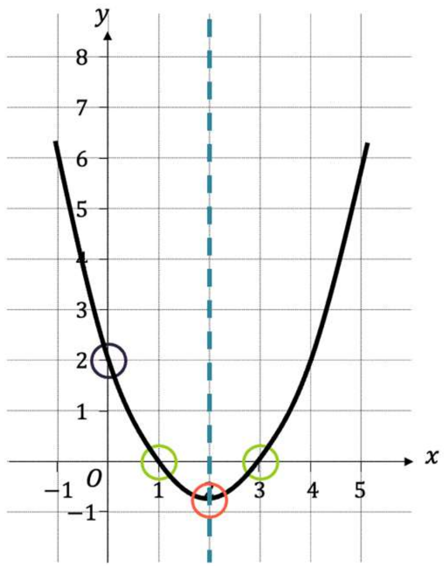
Determine the equation of a quadratic function from its graph only

There are some key features of quadratic graphs that we can use if we want to **sketch** a quadratic rather than plot it.

1 The shape of the graph - ∪ or ∩

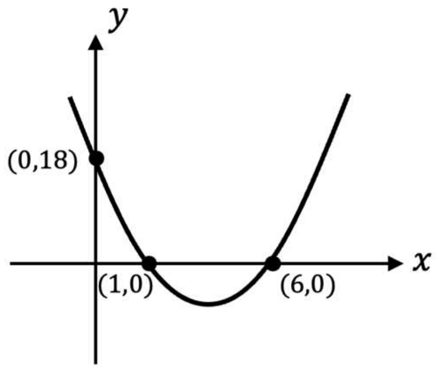
2 Where the graph crosses the y-axis

3 Where the graph crosses the x-axis



4 The line of symmetry of the graph

5 The turning point of the graph



Start by considering where the graph crosses the x-axis:

$x = 6$ and $x = 1$

So, the factorised equation must be:

$$y = a(x-6)(x-1)$$

To find the value of a, we use the coordinate where the graph crosses the y-axis:

$$y = a(x^2 - 7x + 6)$$

When $x = 0, y = 18$

$$18 = a \times 6$$

$$a = 3$$

Finally, we can find the equation:

$$y = 3(x^2 - 7x + 6)$$

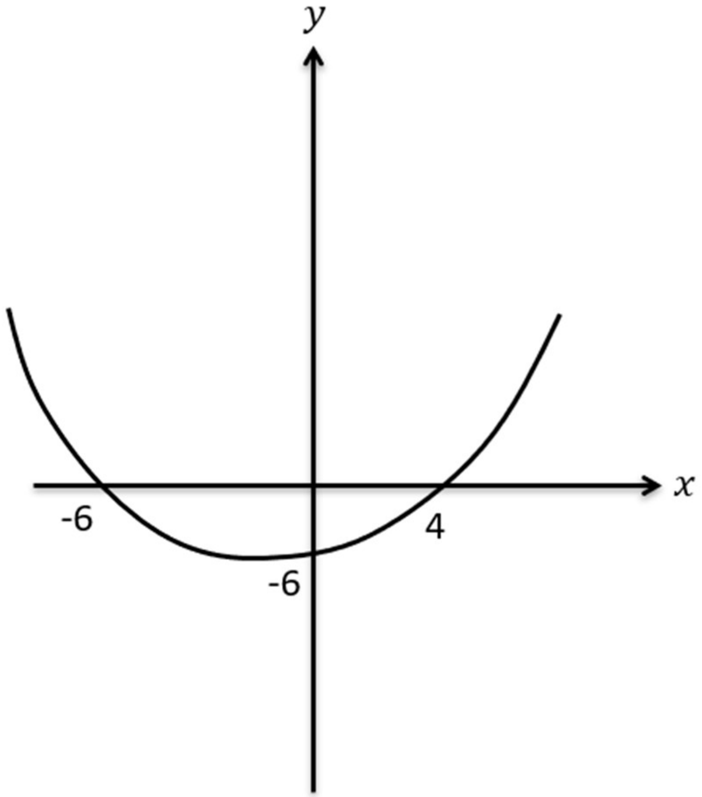
$$\boldsymbol{y = 3x^2 - 21x + 18}$$

Forms of Quadratic Graphs

Form	Standard Form	Vertex Form	Factorised Form
Formula	$y = ax^2 + bx + c$	$y = a(x-g)^2 + h$	$y = a(x-r)(x-s)$
Orientation	a positive: oriented upwards like U. a negative: oriented downwards like ∩.	a positive: oriented upwards like U. a negative: oriented downwards like ∩.	a positive: oriented upwards like U. a negative: oriented downwards like ∩.
Y-intercept	The y-intercept is c.	The y-intercept is found by substituting $x = 0$ into $y = a(x-g)^2 + h$.	The y-intercept is found by substituting $x = 0$ into $y = a(x-r)(x-s)$.
Number of x-intercepts	Calculate the discriminant $b^2 - 4ac$. Discriminant positive: two x-intercepts. Discriminant zero: one x-intercept. Discriminant negative: no x-intercepts.	If h is positive and a is positive: no x-intercepts. If h is positive and a is negative: two x-intercepts. If h is zero: one x-intercept. If h is negative and a is negative: no x-intercepts. If h is negative and a is positive: two x-intercepts.	If $r = s$: one x-intercept. If $r \neq s$: two x-intercepts. (If there are no x-intercepts it cannot be written in this form.)
Roots	Calculate using quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Solve the equation $a(x-g)^2 + h = 0$.	Roots are $x = r$ and $x = s$.
Vertex	The x-coordinate of the vertex is $x = -\frac{b}{2a}$. The y-coordinate is found by substituting this value of x into the quadratic.	The vertex is (g, h).	The x-coordinate of the vertex is halfway between the two roots, at $x = \frac{r+s}{2}$. The y-coordinate is found by substituting this value of x into the quadratic.
Converting to other forms	Converting to standard form from other forms: expand.	Converting to vertex form from standard form: complete the square.	Converting to to factorised form from standard form: factorise.

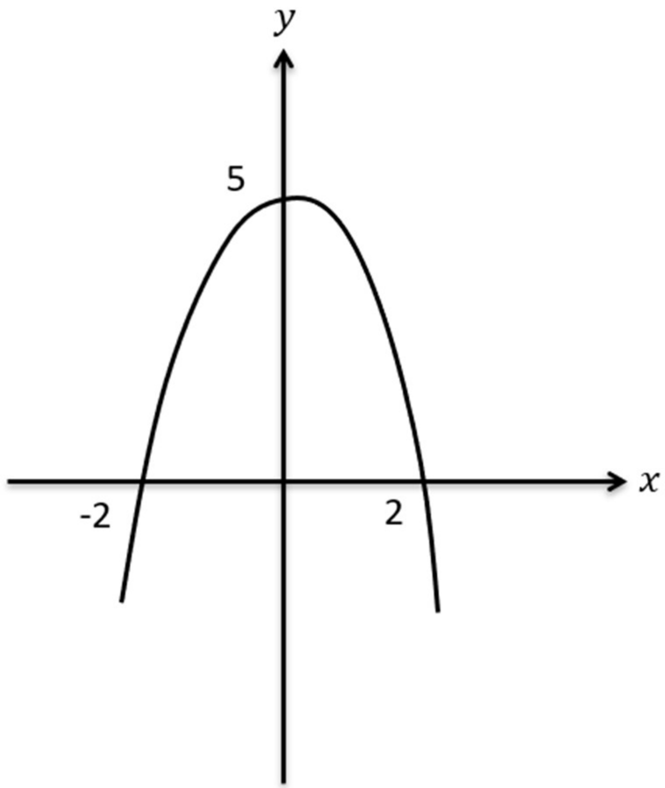
Worked Example

Determine the equation of this quadratic graph, in the form $y = ax^2 + bx + c$.



Worked Example

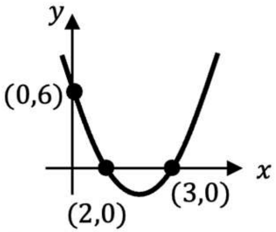
Determine the equation of this quadratic graph, in the form $y = ax^2 + bx + c$, where a, b, c are integers.



Fluency Practice

2 Find the equations of the quadratics from their sketches.

a



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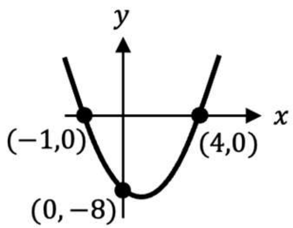
b



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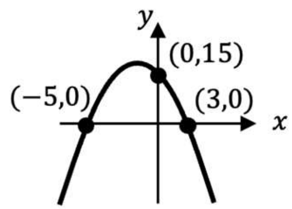
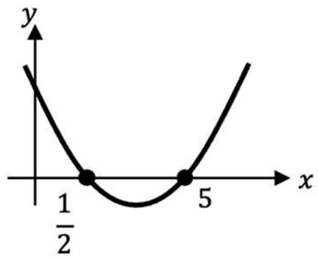
c



?

3 *[AQA IGCSE FM Practice Paper Set 1 P2 Q15]*

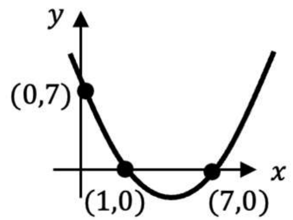
The diagram shows a quadratic graph that intersects the x-axis when $x = \frac{1}{2}$ and $x = 5$.
Work out the equation of the graph.



?

4 Given the sketch of the quadratic, find the equation of the line of symmetry and the coordinates of the turning point.

a



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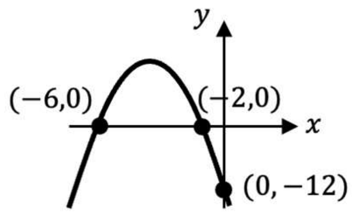
b



?

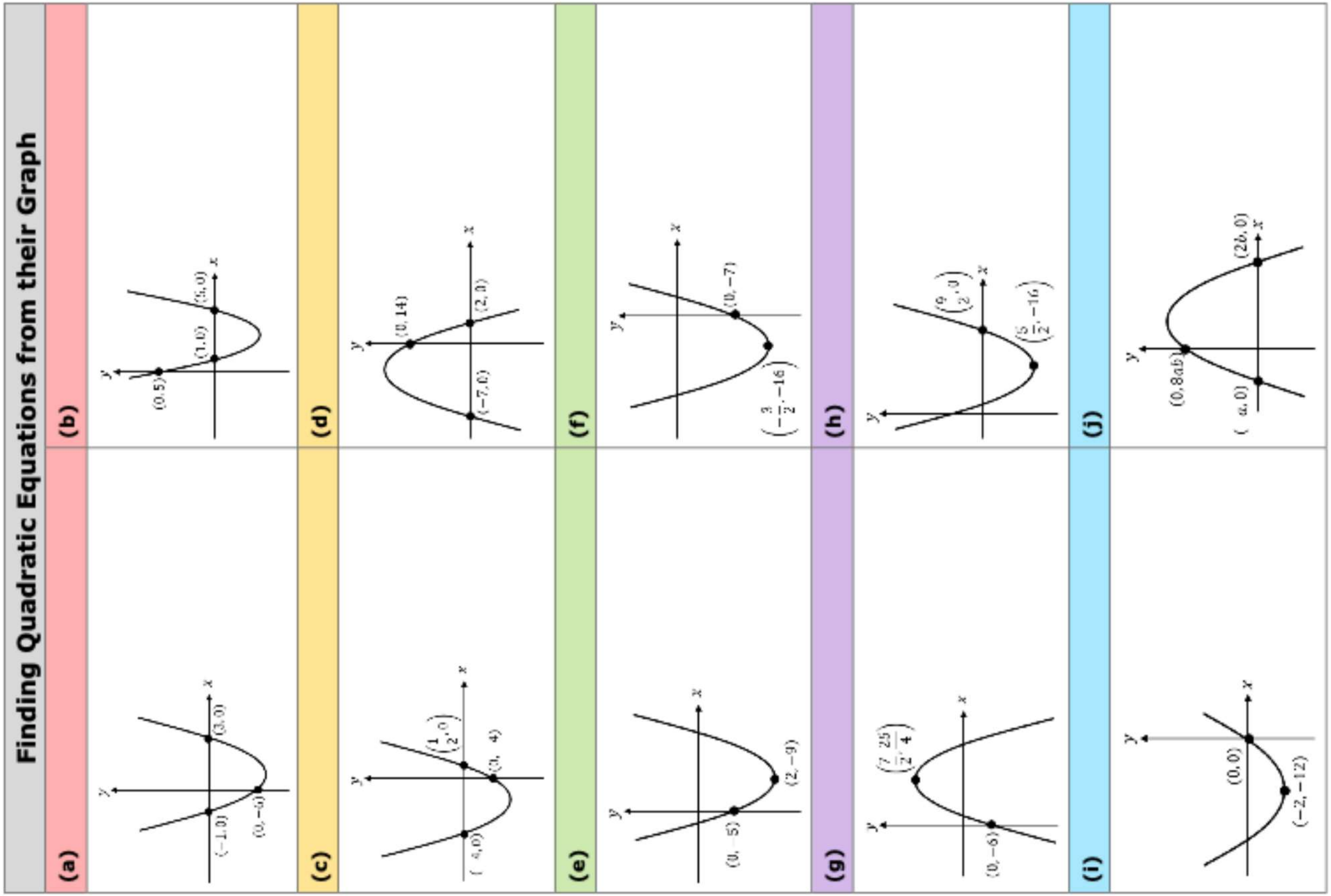
☠ The graph of a quadratic function passes through the point $(-8,0)$ and has a turning point at $(-2,18)$.
Find the equation of the quadratic graph.

?

Worked Example

The graph of $y = ax^2 + bx + c$ has a minimum at $(3, -5)$ and passes through $(4, 0)$. Find the values of a, b and c

Fluency Practice



T.30 2F Qs 2,3 P.10 2.4 Qs 5+

Exam Q

A2 2021 Paper 1

Quadratics

2. Given that

$$f(x) = x^2 - 4x + 5 \qquad x \in \mathbb{R}$$

(a) express f(x) in the form $(x + a)^2 + b$ where a and b are integers to be found.

(2)

The curve with equation $y = f(x)$

- meets the y-axis at the point P
- has a minimum turning point at the point Q

(b) Write down

(i) the coordinates of P

(ii) the coordinates of Q

(2)

Your Turn

2. Given that

$$f(x) = x^2 - 6x + 7 \qquad x \in \mathbb{R}$$

(*a*) express f (*x*) in the form $(x + a)^2 + b$ where *a* and *b* are integers to be found.

(2)

The curve with equation $y = f(x)$

- meets the *y*-axis at the point *P*
- has a minimum turning point at the point *Q*

(*b*) Write down

(i) the coordinates of *P*

(ii) the coordinates of *Q*

(2)

(Total for Question 2 is 4 marks)

1.7 Discriminant

$$ax^2 + bx + c = 0$$
$$x = \frac{-b \pm \sqrt{\boxed{b^2 - 4ac}}}{2a}$$

The term '**root**' refers to a **solution of an equation** in the form $f(x) = 0$, i.e. when one side of the equation is 0.

Reflecting on the quadratic formula you just used, **what part of the formula** do you think **distinguishes** between when we have 0 real roots, 1 distinct real root and 2 distinct real roots?

$$\frac{\ldots \pm \sqrt{\square}}{\ldots}$$

When this was **negative**, we couldn't square root it, so there were no real solutions.

$$\frac{\ldots \pm \sqrt{\square}}{\ldots}$$

When this was **positive**, adding or subtracting the square root gave different, i.e. distinct, solutions.

$$\frac{\ldots \pm \sqrt{\mathbf{0}}}{\ldots}$$

When this was **zero**, adding or subtracting had no effect, so the solution was the same.

✏ The discriminant $\Delta = b^2 - 4ac$ of a quadratic equation $ax^2 + bx + c = 0$ allows us to 'discriminate' between the different **number of roots/solutions**:

$b^2 - 4ac > 0$	Distinct real roots
$b^2 - 4ac < 0$	No real roots
$b^2 - 4ac = 0$	Equal roots

Δ is capital delta in the Greek alphabet. It is unrelatedly also used to mean 'change in', e.g. within the formula for gradient:

$$m = \frac{\Delta y}{\Delta x}$$

Questions

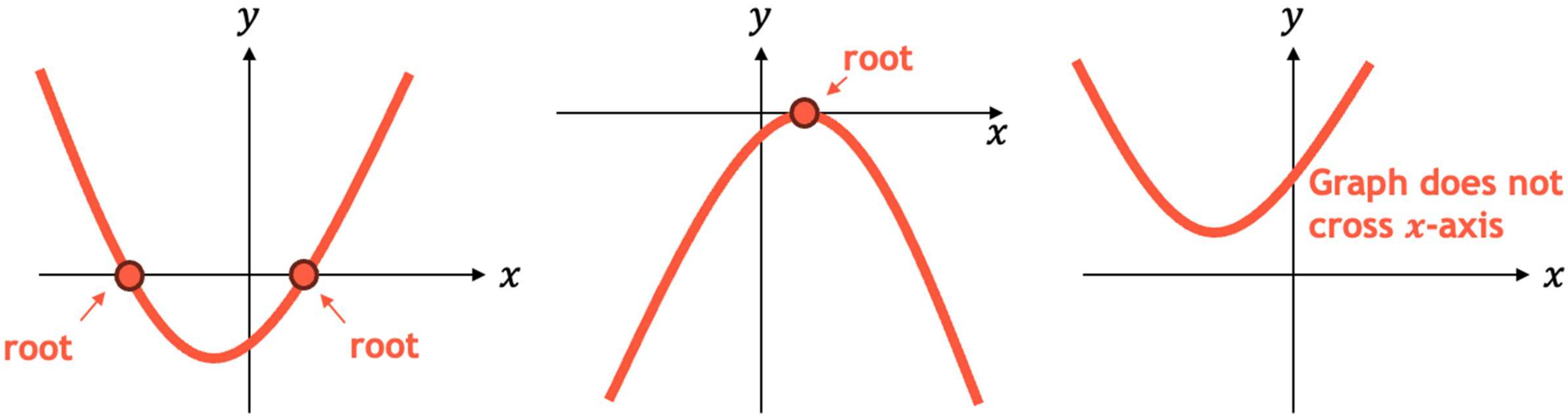
Tip: Write out $a = \cdots, b = \cdots, c = \cdots$ explicitly before calculating the discriminant.

$\Delta = b^2 - 4ac$

Equation	a, b, c	Δ	Number of distinct real roots
$x^2 + 3x + 4 = 0$			
$x^2 - 4x + 1 = 0$			
$x^2 - 4x + 4 = 0$			
$2x^2 - 6x - 3 = 0$			
$x - 4 - 3x^2 = 0$			
$1 - x^2 = 0$			

Quadratic Graphs and the Discriminant

When solving $ax^2 + bx + c = 0$, the solutions correspond to the **<u>x-intercepts/roots</u>** of the graph $y = ax^2 + bx + c$

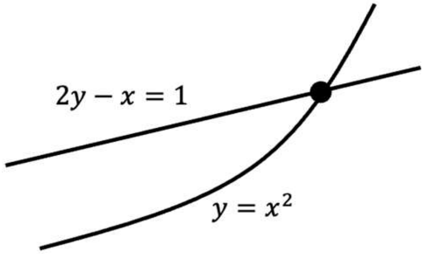


2 distinct real roots so: $\Delta > 0$

1 distinct root so: $\Delta = 0$

0 real roots so: $\Delta < 0$

Using the Discriminant for Points of Intersection



How would you normally determine the points at which two lines, with given equations, intersect?

The **<u>points of intersection</u>** **<u>correspond to the solutions</u>** to the system of equations:

$$y = x^2$$
$$2y - x = 1$$

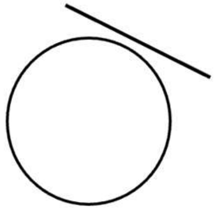
We could solve this by substitution:

$$2x^2 - x = 1$$

The number of solutions to this equation therefore corresponds to the number of points of intersection.
We can use the discriminant to determine this.

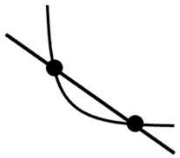
If a line is **<u>tangent to a curve</u>**, then there will be 1 point of intersection, and therefore **<u>1 distinct root</u>** to the equation obtained by substitution.

$$b^2 - 4ac = 0$$



Similarly, **<u>if the line never meets the curve</u>**, then the equation obtained by substitution will have no real roots.

$$b^2 - 4ac < 0$$



How many solutions would you expect there be to the equation obtained by substitution of this line and curve?

There will be 2 points of intersection, and therefore **<u>2 distinct roots.</u>**

$$b^2 - 4ac > 0$$

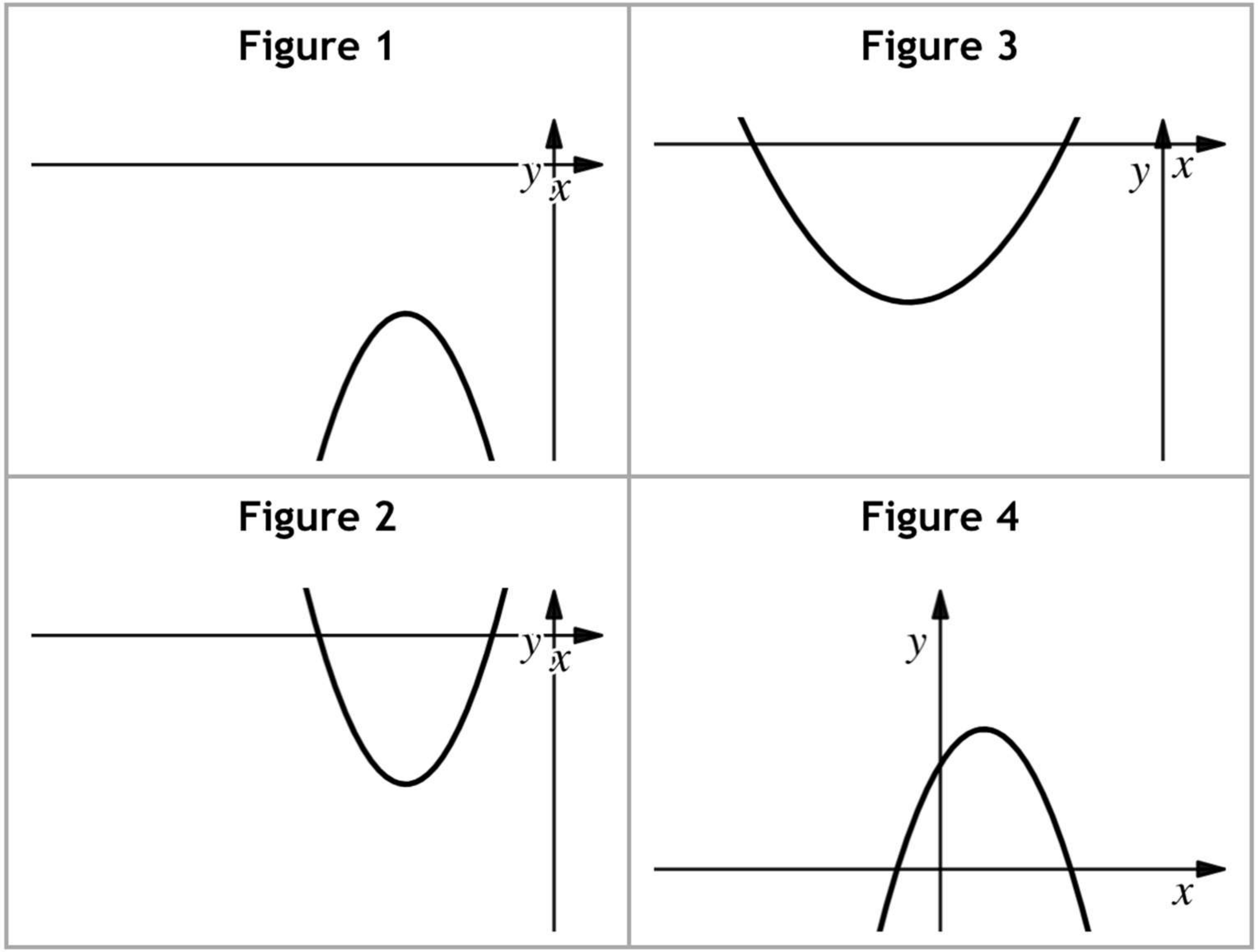
Notes

Fill in the Gaps

Quadratic Equation	a	b	c	Discriminant	Nature of Solutions
$x^2 + 5x + 3 = 0$	1	5	3	$5^2 - 4 \times 1 \times 3 = 13$	Two real solutions
$3x^2 - 2x + 5 = 0$	3	−2	5		
$x^2 + 4x + 4 = 0$					
$6x^2 + 5 = 0$					
$16 - x - 3x^2 = 0$					
$9x^2 - 2x = 0$					
	5	4	1		
	1	8		44	
		−9	8	−111	
		−12	9		One real solution
$5x^2 + 7x - 3 = 0$	5	7		109	
	−1		−25		One real solution
		6	5	−24	

Worked Example

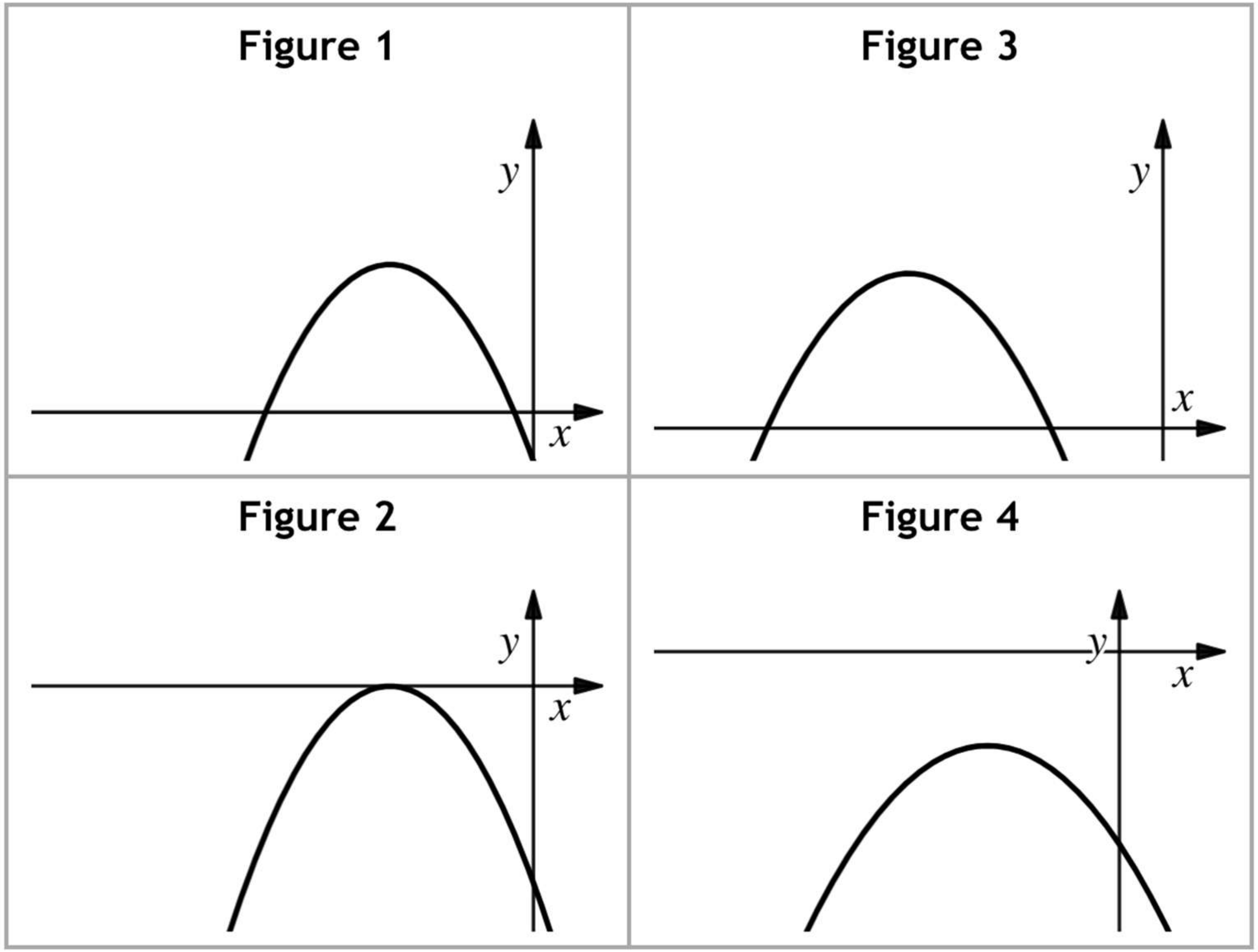
Calculate the discriminant and, hence, identify which figure represents the graph with equation $y = -4x^2 + 6x + 7$



Dr Frost 492c

Your Turn

Calculate the discriminant and, hence, identify which figure represents the graph with equation $y = -2x^2 - 5x - 6$



Dr Frost 492c

Fluency Practice

	Factorises (with integers)	**Does not factorise**
Has two real and different roots $b^2-4ac>0$ Graph crosses the x-axis		
Has one real root (2 identical roots) $b^2-4ac=0$ Graph just touches the x-axis		***If the quadratic equation has one root then it always factorises***
Has no real roots $b^2-4ac<0$ Graph does not touch or cross the x-axis	***If the quadratic equation has no real roots then it never factorises***	

$x^2-8x+16=0$	$3x^2+7x-6=0$	$x^2+2x+4=0$	$x^2-25=0$
$x^2+3x-11=0$	$x^2+7x+12=0$	$-x^2-8x+2=0$	$-x^2+x-2=0$
$-2x^2+9x-4=0$	$x^2-2x+1=0$	$6x^2+11x-10=0$	$x^2-3x+6=0$
$2x^2+10x-14=0$	$2x^2-12x+18=0$	$x^2+4x+4=0$	$x^2+6x+10=0$
$-2x^2+4x-6=0$	$x^2+6x+4=0$	$3x^2+60x+300=0$	$2x^2+3x-1=0$

Fluency Practice

	Factorises	**Does not factorise**
Has two real and different roots $b^2-4ac>0$ Graph crosses the x-axis		
Has one real root (2 identical roots) $b^2-4ac=0$ Graph just touches the x-axis		*If the quadratic equation has one root then it always factorises*
Has no real roots $b^2-4ac<0$ Graph does not touch or cross the x-axis	*If the quadratic equation has no real roots then it never factorises*	

$x^2-p^2=0$	$x^2+p^2=0$	$x^2-2px+p^2=0$	$q^2x^2+p^2=0$
$x^2+2px+p^2=0$	$x^2+(p+q)x+pq=0$	$qx^2+2pqx+qp^2=0$	$q^2x^2-p^2=0$
$x^2+(p-q)x-pq=0$	$x^2-(p+q)x+pq=0$	$q^2x^2-p^2=0$	$q^2x^2-2pqx+p^2=0$

$p,q \in \sim$ and $p,q \neq 0$

Worked Example

The straight line with equation $y = 7px + 9p$ touches the curve with equation $y = 6px^2 - x + 3$, where p is a constant. Find the set of possible values of p

Worked Example

Prove that the function $f(x) = 4x^2 + (k+8)x - k$ has two distinct real roots for all values of k

T. 32 2G Qs 2+, P.11 2.5 Qs 4+

Extension

5 *[MAT 2009 1C]*

Given a real constant c, the equation

$$x^4 = (x-c)^2$$

Has four real solutions (including possible repeated roots) for:

A) $c \leq \frac{1}{4}$

B) $-\frac{1}{4} \leq c \leq \frac{1}{4}$

C) $c \leq -\frac{1}{4}$

D) all values of c

?

6 *[MAT 2006 1B]*

The equation $(2+x-x^2)^2 = 16$ has how many real root(s)?

?

7 *[MAT 2011 1B]*

A rectangle has perimeter P and area A. The values P and A must satisfy:

A) $P^3 > A$

B) $A^2 > 2P + 1$

C) $P^2 \geq 16A$

D) $PA > A + P$

?

1.8 Modelling with Quadratics

Worked Example

A spear is thrown over level ground from the top of a tower.
The height, in metres, of the spear above the ground after t seconds is modelled by the function: $h(t) = 1.65 + 24.5t - 4.9t^2$, $t \geq 0$

a) Interpret the meaning of the constant term 12.25 in the model.
b) After how many seconds does the spear hit the ground?
c) Write $h(t)$ in the form $A - B(t - C)^2$, where A, B and C are constants to be found.
d) Using your answer to part c or otherwise, find the maximum height of the spear above the ground, and the time at which this maximum height is reached?

T.34 2H Qs all, P.11 2.6 Qs all

A2 2021 Paper 1

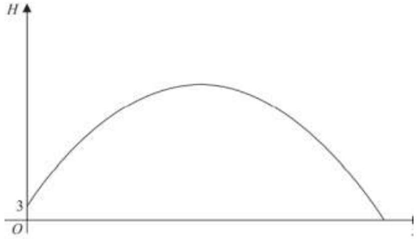


Figure 3

Figure 3 is a graph of the trajectory of a golf ball after the ball has been hit until it first hits the ground.

The vertical height, H metres, of the ball above the ground has been plotted against the horizontal distance travelled, x metres, measured from where the ball was hit.

The ball is modelled as a particle travelling in a vertical plane above horizontal ground.

Given that the ball

- is hit from a point on the top of a platform of vertical height 3 m above the ground
- reaches its maximum vertical height after travelling a horizontal distance of 90 m
- is at a vertical height of 27 m above the ground after travelling a horizontal distance of 120 m

Given also that H is modelled as a **quadratic** function in x

(a) find H in terms of x

(5)

(b) Hence find, according to the model,

(i) the maximum vertical height of the ball above the ground,

(ii) the horizontal distance travelled by the ball, from when it was hit to when it first hits the ground, giving your answer to the nearest metre.

(3)

(c) The possible effects of wind or air resistance are two limitations of the model. Give one other limitation of this model.

(1)

Your Turn

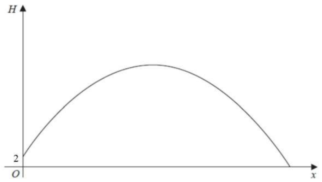


Figure 3

Figure 3 is a graph of the trajectory of a golf ball after the ball has been hit until it first hits the ground.

The vertical height, H metres, of the ball above the ground has been plotted against the horizontal distance travelled, x metres, measured from where the ball was hit.

The ball is modelled as a particle travelling in a vertical plane above horizontal ground.

Given that the ball

- is hit from a point on the top of a platform of vertical height 2 m above the ground
- reaches its maximum vertical height after travelling a horizontal distance of 100 m
- is at a vertical height of 34 m above the ground after travelling a horizontal distance of 160 m

Given also that H is modelled as a **quadratic** function in x

(*a*) find H in terms of x

(5)

(*b*) Hence find, according to the model,

(i) the maximum vertical height of the ball above the ground,

(ii) the horizontal distance travelled by the ball, from when it was hit to when it first hits the ground, giving your answer to the nearest metre.

(3)

(*c*) The possible effects of wind or air resistance are two limitations of the model. Give one other limitation of this model.

(1)

(Total for Question 12 is 9 marks)

Summary

1 To solve a quadratic equation by factorising:

- Write the equation in the form $ax^2 + bx + c = 0$
- Factorise the left-hand side
- Set each factor equal to zero and solve to find the value(s) of x

2 The solutions of the equation $ax^2 + bx + c = 0$ where $a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3 $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

4 $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$

5 The set of possible inputs for a function is called the **domain**.

The set of possible outputs of a function is called the **range**.

6 The **roots** of a function are the values of x for which $f(x) = 0$.

7 You can find the coordinates of a **turning point** of a quadratic graph by completing the square. If $f(x) = a(x + p)^2 + q$, the graph of $y = f(x)$ has a turning point at $(-p, q)$.

8 For the quadratic function $f(x) = ax^2 + bx + c = 0$, the expression $b^2 - 4ac$ is called the **discriminant**. The value of the discriminant shows how many roots $f(x)$ has:

- If $b^2 - 4ac > 0$ then a quadratic function has two distinct real roots.
- If $b^2 - 4ac = 0$ then a quadratic function has one repeated real root.
- If $b^2 - 4ac < 0$ then a quadratic function has no real roots.

9 Quadratics can be used to model real-life situations.

1.9 Simultaneous Equations (linear, quadratic and graphical)

Prior knowledge check

3.1 Linear simultaneous equations

1 Solve these simultaneous equations by elimination:

a $6x + y = 9$
$4x - y = 11$

b $2x + 3y = 8$
$3x + 2y = 7$

c $4x - 3y = 2$
$5x - 7y = 9$

Hint Either the x- or y-coefficients in the pair of equations need to have the same value.
You can then add or subtract the equations to eliminate one variable and solve for the other.
You need to find the value of both variables.

2 Solve these simultaneous equations by substitution:

a $5x - 2y = 3$
$x + 4y = 5$

b $2x + 5y = 37$
$y = 11 - 2x$

c $4x + 3y = 5$
$2x - 6y = -5$

Hint Rearrange one equation to make x or y the subject and then substitute this expression into the other equation and solve.

3 Solve these simultaneous equations:

a $3(x - y) + 6 = 0$
$y + x = 8$

b $5(x - 1) = -2y$
$3x - 29 = 4y$

c $\frac{4x - y}{2} = 11$
$\frac{5 - 3x}{5} = y$

Hint Rearrange the equations into the form $ax + by = c$, where a, b and c are constants, and then solve using either elimination or substitution.

(E) 4 Solve the simultaneous equations
$x + y = 2$
$2y = 18x - 6$ **(2 marks)**

(E) 5 Solve the simultaneous equations
$3x - y = -5$
$0.5y + 2x = 4$
giving your answers in exact form. **(2 marks)**

(E/P) 6 $6ky + 9x = 12$
$ky - x = 4.5$
are simultaneous equations where k is a constant.

a Show that $x = -1$. **(2 marks)**

b Given that $y = 7$, find the value of k. **(1 mark)**

(E/P) 7 Two students are attempting to solve the simultaneous equations
$4x + 6y = 10$
$2x = 5 - 3y$
Ben says that these equations have no solutions, and Nisha says that they have infinitely many solutions. Who is correct? Explain your answer. **(2 marks)**

3.2 Quadratic simultaneous equations

1 Solve the simultaneous equations:

a $xy = 64$
$4x - y = 60$

b $x^2 + y^2 = 10$
$x + y = 4$

c $x - y + 1 = 0$
$3x^2 - 4y = 0$

Hint A quadratic equation can contain terms involving xy, x^2 and y^2.
Rearrange the linear equation to make x or y the subject, then substitute this expression into the quadratic equation and solve.

2 Solve the simultaneous equations:

a $y - x^2 + 3x + 2 = 0$
$y - 2x + 6 = 0$

b $x^2 + 2x - y = 14$
$y + 2 = x$

c $x^2 + y^2 = 5$
$y = 5 - 3x$

Hint Each set of equations will have two pairs of solutions.
You need to identify the solutions and pair them up correctly.

3 Solve these simultaneous equations, giving your answers to 2 decimal places:

a $3x - 7 = y$
$x^2 - 3x - 2 = 2y$

b $2x^2 - xy + y^2 = 8$
$x + y = 1$

c $y^2 - 5x^2 = 20$
$4x - 7 = y$

(E) 4 Solve the simultaneous equations
$x + y = 4$
$4y^2 - x^2 = 12$ **(4 marks)**

(E) 5 Solve the simultaneous equations
$y + 2x + 1 = 0$
$y^2 + 3x^2 + 2x = 0$ **(4 marks)**

(E/P) 6 $mx - y - 2 = 0$
$x^2 - 2x + y^2 - 4y = 4$
where m is a real constant.

Given that these simultaneous equations have exactly one pair of solutions, find the two possible values of m, giving your answers in exact form. **(7 marks)**

(E/P) 7 The values of x and y satisfy the simultaneous equations
$y - 2x = 8$
$x^2 + 2ky + 4k = 0$
where k is a non-zero constant.

a Show that $x^2 + 4kx + 20k = 0$. **(2 marks)**

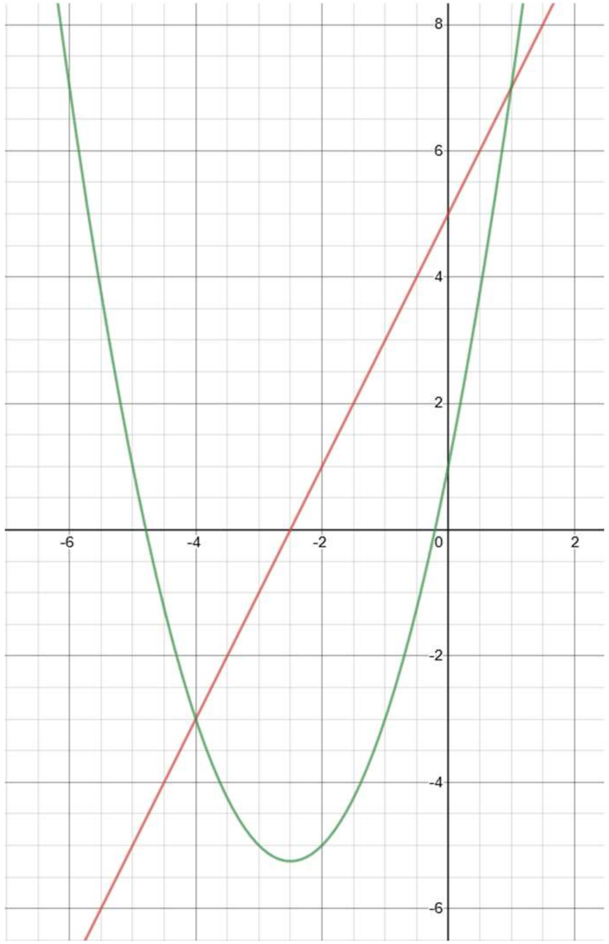
b Given that $x^2 + 4kx + 20k = 0$ has equal roots, find the value of k. **(3 marks)**

c For the value of k found in part **b**, solve the simultaneous equations. **(3 marks)**

Worked example

Solve:

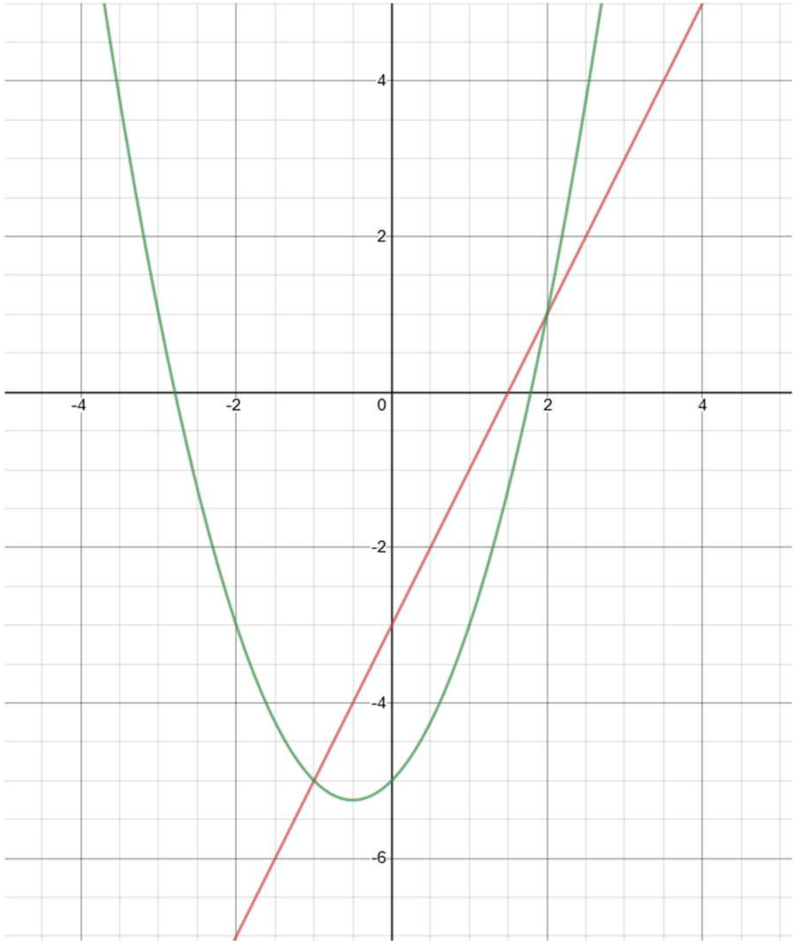
$$y = 2x + 5$$
$$y = x^2 + 5x + 1$$



Your turn

Solve:

$$y = 2x - 3$$
$$y = x^2 + x - 5$$



T.45 : 3C Qs 1-4, P.18: 3.3 all Qs 1-4

Worked Example

Prove algebraically, and show graphically, that the lines never meet:

$$y = 3x - 3$$
$$y = x^2 + 5x + 4$$

T.45 : 3C Qs 5-7, P.18: 3.3 all Qs 5-6

Worked Example

The line with equation $y = 3x + 4$ meets the curve with equation $kx^2 + 2y + (k - 8) = 0$ at exactly one point. Given that k is a positive constant:

a) Find the value of k.

b) For this value of k, find the coordinates of this point of intersection.

T.45 : 3C Qs 8+, P.18: 3.3 all Qs 7+

1.10 Inequalities (linear, quadratic and graphical, including set notation)

Recall that a **set** is a **collection of values** such that:

a) The **order of values does not matter**.

b) There are **no duplicates**.

Recap from GCSE:

- We use curly braces to list the values in a set, e.g. $A = \{1,4,6,7\}$
- If A and B are sets then $A \cap B$ is the **intersection** of A and B, giving a set which has the elements in A **and** B.
- $A \cup B$ is the **union** of A and B, giving a set which has the elements in A **or** in B.
- $\emptyset$ is the empty set, i.e. the set with nothing in it.
- Sets can also be infinitely large. $\mathbb{N}$ is the set of natural numbers (all positive integers), $\mathbb{Z}$ is the set of all integers (including negative numbers and 0) and $\mathbb{R}$ is the set of all real numbers (including all possible decimals).
- We write $x \in A$ to mean "x is a member of the set A". So $x \in \mathbb{R}$ would mean "x is a real number".

$$\{1,2,3\} \cap \{3,4,5\} = \{\mathbf{3}\}$$
$$\{1,2,3\} \cup \{3,4,5\} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$$
$$\{1,2\} \cap \{3,4\} = \emptyset$$

Set Builder Notation

It is possible to construct sets without having to explicitly list its values. We use:

The | or : means "such that".

$$\{expr \mid \; condition\}$$

or $$\{expr : \; condition\}$$

Can you guess what sets the following give?

(In words "All numbers $2x$ such that x is an integer)

$$\{2x : x \in \mathbb{Z}\} = \{0,2,-2,4,-4,6,-6,\ldots\}$$

i.e. The set of all even numbers!

$$\{2^x : x \in \mathbb{N}\} = \{2,4,8,16,32,\ldots\}$$

$$\{xy : \; x, y \; are \; prime\} = \{4,6,10,14,15,\ldots\}$$

i.e. All possible products of two primes.

We previously talked about 'solutions sets', so set builder notation is very useful for specifying the set of solutions!

Examples

All odd numbers.

$$\{2x + 1 \ : \ \ x \in \mathbb{Z}\}$$

All (real) numbers greater than 5.

$$\{x: \ \ x > 5\}$$

Technically it should be $\{x: \ \ x > 5, x \in \mathbb{R}\}$ but the $x > 5$ by default implies **real numbers** greater than 5.

All (real) numbers less than 5 **or** greater than 7.

$$\{x: \ \ x < 5\} \cup \{x: x > 7\}$$

We combine the two sets together.

All (real) numbers between 5 and 7 inclusive.

$$\{x: \ \ 5 \leq x \leq 7\}$$

While we could technically write $\{x: x \geq 5\} \cap \{x: x \leq 7\}$, we tend to write multiple required conditions within the same set.

Notes

Fill in the blanks

Inequality/Inequalities	In Set Notation
$x \geq 3.2$	$\{x: x \geq 3.2\}$
$-2 \leq x < 5$	
	$\{x: x < -8\} \cup \{x: x \geq 3\}$
$-\frac{1}{2} < x < 4$	$\{\quad\} \cap \{\quad\}$
	$\left\{x: -\frac{1}{2} < x < 4\right\}$
$x < \frac{7}{2}$ and $x > 1$ $x \leq 1$ or $x \geq 6.5$	
$-\frac{3}{4} \leq x \leq -\frac{1}{4}$	
	$\{x: x \leq -3\} \cup \{x: 2 < x < 5\}$
$x > 10$ or $-1 \leq x < 0$	
	$\{x: x \geq 5\} \cup \left\{x: x < \frac{3}{2}\right\}$
	$\left\{x: -2 \leq x < \frac{3}{5}\right\} \cup \{x: x > 1\}$
$x < 9$ and $x > \frac{2}{3}$	

Worked Example

Use set notation to describe the set of values for which:

$$10(9x+8) < 7 \text{ or } 6(5x-4) \geq \frac{3-2x}{4}$$

T.48: 3D Q3 , P.19: 3.4 Q5-8

Quadratic inequalities

Recall you need to avoid multiplying (or dividing) by a negative. **This includes a variable $\boldsymbol{x}$** *(as it could be negative).*

E.g. $\frac{a}{x} < b$ **becomes** $ax < bx^2$

You need to remember how to use dotted and solid line convention for inequalities, i.e.

$> or\ <$ go with dotted (or hollow circle)
$\geq or\ \leq$ with solid (or shaded circle)

Notes

Worked Example

458k: **Solve quadratic inequalities given in the form $-x^2 + bx + c > 0$ or $-x^2 + bx + c < 0$**

Solve:

$$5k - 6 - k^2 \leq 0$$

T.50: 3E Qs1-3 , P.20: 3.5 Qs 1-2

Fill in the blanks

First Inequality	Solution to 1st Inequality	2nd Inequality	Solution to 2nd Inequality	Combined Solution
$2x - 1 \geq 4x + 3$		$3(7 + x) > 6$		$-5 < x \leq -2$
$\frac{3 + 2x}{4} < 3$		$5x - 2 < 5 + x$		
$3(x - 3) > 2x - 11$		$5 - x \leq 7 - 4x$		
$7(2x + 1) > 4x - 3$		$x^2 - 2x - 8 \leq 0$		
$x^2 - 7x + 10 < 0$		$6(1 - x) \leq 0$		
$\frac{5x + 1}{8} > 2$		$2x^2 - 9x \geq 5$		
$2(2 - 3x^2) \geq 5x$		$12 - 5x < 9 - 2x$		
$2x^2 < 50$		$5x(x - 6) \leq 8x - 21$		

Worked Example

Find the set of values for which $\frac{10}{x} > 5, x \neq 0$

Worked Example

Find the set of values for which $\frac{5}{x-3} < 2$

T.50: 3E Q 4 , P.20: 3.5 Q3

Worked Example

458m: Solve quadratic inequalities involving a division by a positive algebraic expression.

Solve

$$1 < \frac{7x}{2x^2 + 3}$$

Worked Example

4580: Combine a linear and a quadratic inequality.

n is an integer such that $7n - 4 \geq 24$ and $\dfrac{5n}{n^2+4} \geq 1$

Find all the possible values of n.

Worked Example

The equation $kx^2 - 5kx + 50 = 0$, where k is a constant, has no real roots.
Prove that k satisfies the inequality $0 \leq k < 8$

T.50: 3E Qs4+, P.20: 3.5 Qs 4+

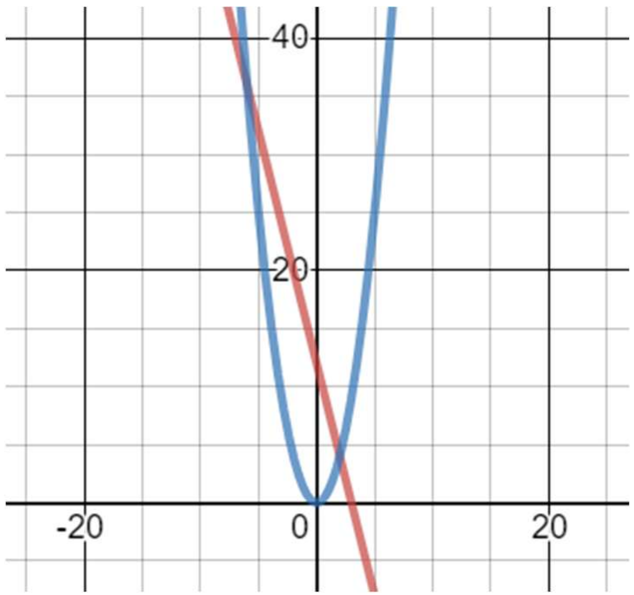
Inequalities on graphs

Worked Example

L_1 has equation $y = 12 - 4x$.
L_2 has equation $y = x^2$.
The diagram shows a sketch of L_1 and L_2 on the same axes.

a) Find the coordinates of the points of intersection.

b) Hence write down the solution to the inequality $12 - 4x > x^2$



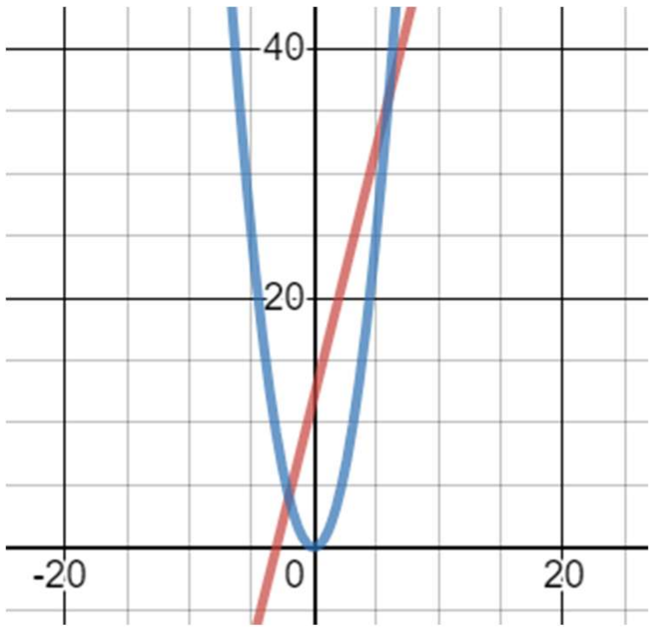
T.52: 3F all Q+, P.21: 3.6 all Qs

Your Turn

L_1 has equation $y = 12 + 4x$.
L_2 has equation $y = x^2$.
The diagram shows a sketch of L_1 and L_2 on the same axes.

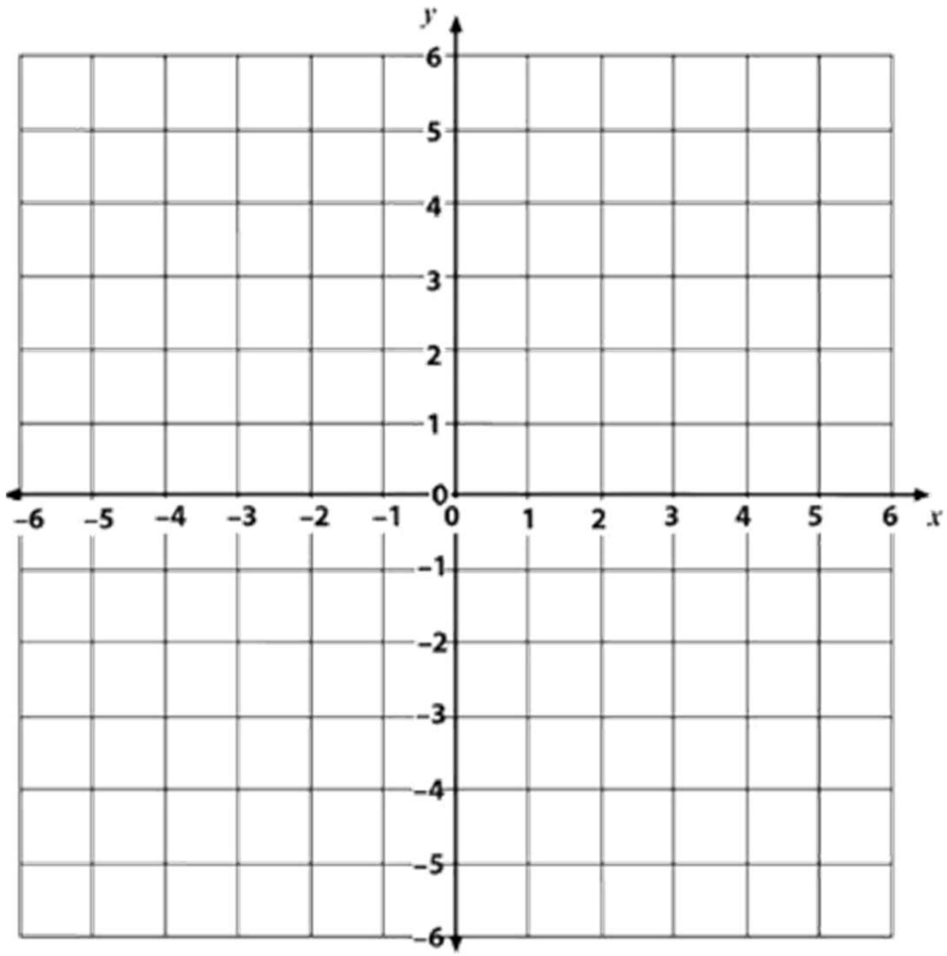
a) Find the coordinates of the points of intersection.

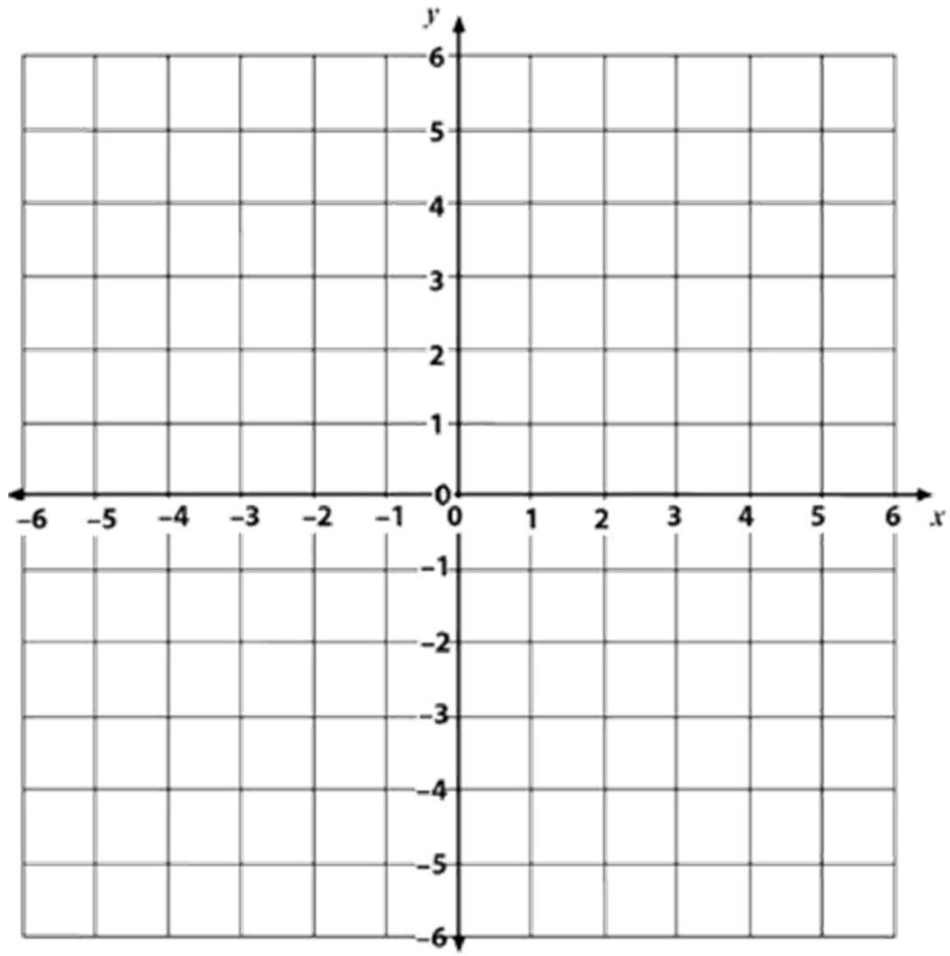
b) Hence write down the solution to the inequality $12 + 4x > x^2$



3.7) Regions

Worked example	Your turn
Shade the region which satisfies the inequalities: $x \geq -2, y < 1$ and $y < x - 1$	Shade the region which satisfies the inequalities. Label it R. $x > -3, y \leq 4$ and $y < x - 2$

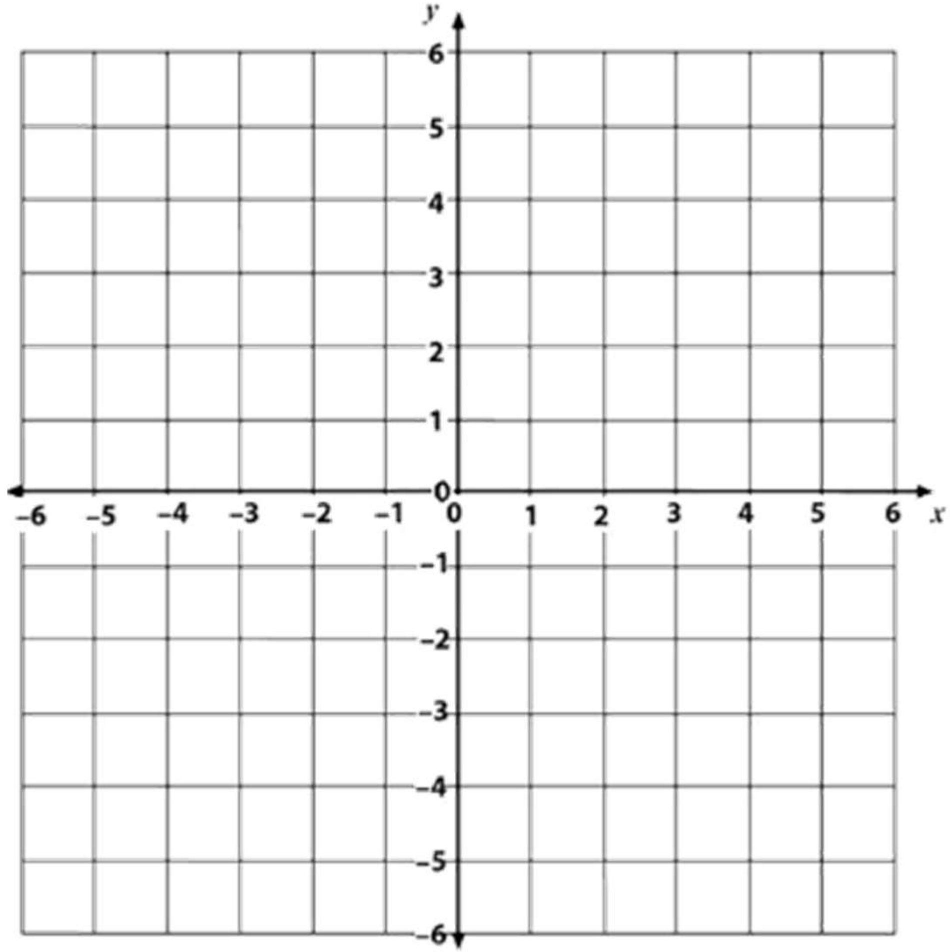




Worked example

Shade the region which satisfies the inequalities:

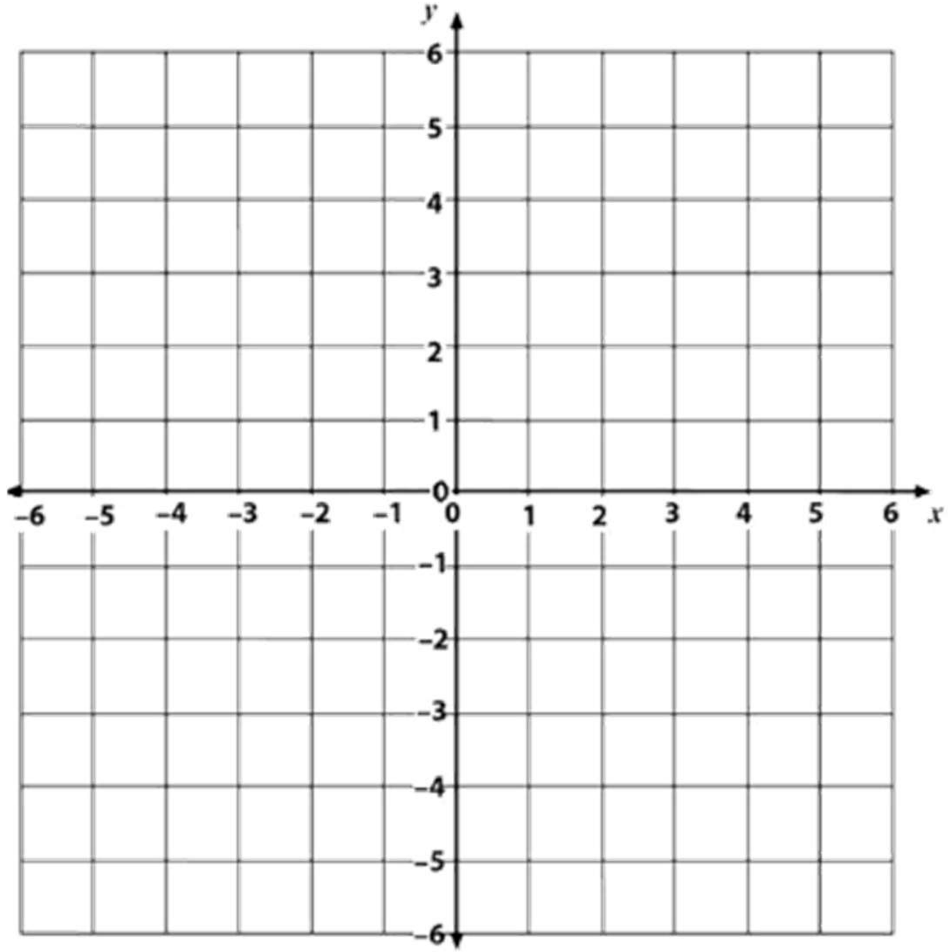
$x \geq 2, y > -1$ and $x + y \leq 5$



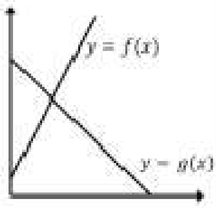
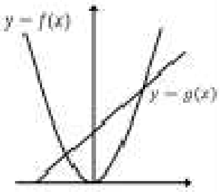
Your turn

Shade the region which satisfies the inequalities. Label it R.

$x \geq 2, y > 1$ and $x + y \leq 6$



Fill in the blanks

$f(x)$ and $g(x)$	Sketch of $y = f(x)$ and $y = g(x)$	Coordinates of intersection(s)	Solutions to $f(x) \geq g(x)$	Shade the region given by:
$f(x) = 3x + 1$ $g(x) = 5 - x$				$y \geq 3x + 1$ $y \leq 5 - x$ and $x \geq 0$
$f(x) = 2x - \frac{1}{2}$ $g(x) = \frac{1}{3}x + 2$				$y \geq 2x - \frac{1}{2}$ $y \leq \frac{1}{3}x + 2$ $x \geq 0$ and $y \geq 0$
$f(x) = x^2$ $g(x) = x + 6$				$y \geq x^2$ and $y \leq x + 6$
$f(x) = 1 - x^2$ $g(x) = x - 1$				$y \leq 1 - x^2$ and $y \leq x - 1$
$f(x) = 4 - 2x$ $g(x) = x^2 + x - 6$				$y \leq 4 - 2x$ and $y \leq x^2 + x - 6$
$f(x) = 4 - x^2$ $g(x) = 2x^2 + 4x$				$y \leq 4 - x^2$ and $y \geq 2x^2 + 4x$

Worked Example

Shade the region that satisfies the inequalities:

$$4y + x \leq 12$$
$$y > x^2 - 5x - 6$$

1. **In this question you should show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

Using algebra, solve the inequality

$$x^2 - x > 20$$

writing your answer in set notation.

(3)

Your Turn

1. **In this question you should show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

Using algebra, solve the inequality

$$x^2 - x > 30$$

writing your answer in set notation.

(3)

(Total for Question 1 is 3 marks)

7.

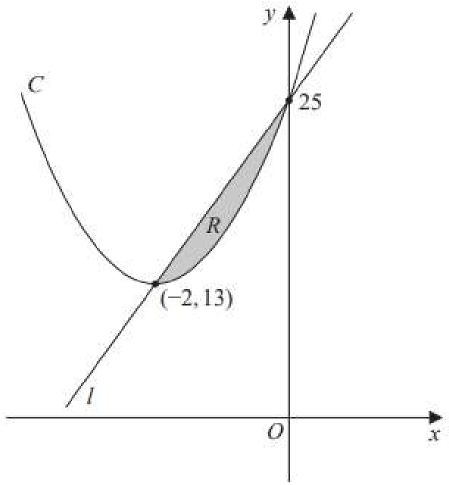


Figure 1

Figure 1 shows a sketch of a curve C with equation $y = \mathrm{f}(x)$ and a straight line l.

The curve C meets l at the points $(-2, 13)$ and $(0, 25)$ as shown.

The shaded region R is bounded by C and l as shown in Figure 1.

Given that

- $\mathrm{f}(x)$ is a quadratic function in x
- $(-2, 13)$ is the minimum turning point of $y = \mathrm{f}(x)$

use inequalities to define R.

(5)

Your Turn

7.

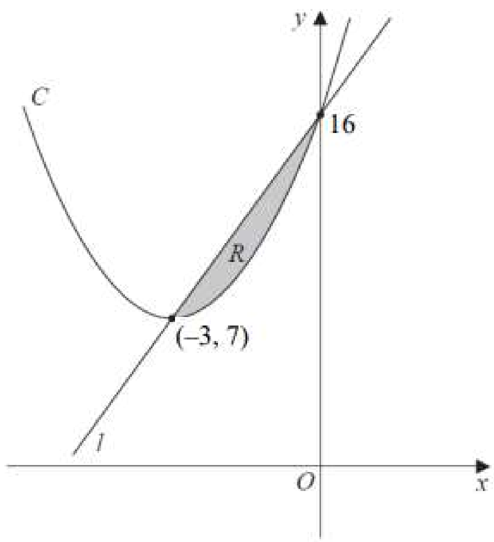


Figure 1

Figure 1 shows a sketch of a curve C with equation $y = \mathrm{f}(x)$ and a straight line l.

The curve C meets l at the points $(-3, 7)$ and $(0, 16)$ as shown.

The shaded region R is bounded by C and l as shown in Figure 1.

Given that $\mathrm{f}(x)$ is a quadratic function in x and that $(-3, 7)$ is the minimum turning point of $y = \mathrm{f}(x)$, use inequalities to define R.

(Total for Question 7 is 5 marks)

1. **In this question you should show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

Using algebra, solve the inequality

$$x^2 - x > 20$$

writing your answer in set notation.

(3)

Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

Question	Scheme	Marks	AOs
1	Finds critical values $x^2 - x > 20 \Rightarrow x^2 - x - 20 > 0 \Rightarrow x = (5, -4)$	M1	1.1b
	Chooses outside region for their values Eg. $x > 5, x < -4$	M1	1.1b
	Presents solution in set notation $\{x : x < -4\} \cup \{x : x > 5\}$ oe	A1	2.5
		(3)	
	(3 marks)		

Summary of Key Points

Summary of key points

1 Linear simultaneous equations can be solved using elimination or substitution.

2 Simultaneous equations with one linear and one quadratic equation can have up to two pairs of solutions. You need to make sure the solutions are paired correctly.

3 The solutions of a pair of simultaneous equations represent the points of intersection of their graphs.

4 For a pair of simultaneous equations that produce a quadratic equation of the form $ax^2 + bx + c = 0$:

- $b^2 - 4ac > 0$ two real solutions
- $b^2 - 4ac = 0$ one real solution
- $b^2 - 4ac < 0$ no real solutions

5 The solution of an inequality is the set of all real numbers x that make the inequality true.

6 To solve a quadratic inequality:

- Rearrange so that the right-hand side of the inequality is 0
- Solve the corresponding quadratic equation to find the critical values
- Sketch the graph of the quadratic function
- Use your sketch to find the required set of values.

7 The values of x for which the curve $y = f(x)$ is **below** the curve $y = g(x)$ satisfy the inequality $f(x) < g(x)$.

The values of x for which the curve $y = f(x)$ is **above** the curve $y = g(x)$ satisfy the inequality $f(x) > g(x)$.

8 $y < f(x)$ represents the points on the coordinate grid below the curve $y = f(x)$.

$y > f(x)$ represents the points on the coordinate grid above the curve $y = f(x)$.

9 If $y > f(x)$ or $y < f(x)$ then the curve $y = f(x)$ is not included in the region and is represented by a dotted line.

If $y \geqslant f(x)$ or $y \leqslant f(x)$ then the curve $y = f(x)$ is included in the region and is represented by a solid line.

1.11 Algebraic division and the Factor Theorem

$$\frac{11}{4} = 2\frac{3}{4} \quad \text{or} \quad 2 \; remainder \; 3$$

dividend (the thing we're dividing)

divisor (the thing we're dividing by)

quotient

$$\frac{f(x)}{divisor(d)} = quotient\ (q) + \frac{remainder\ (r)}{divisor\ (d)}$$

$quotient\ (q)$

$divisor\ (d)$ | $dividend\ f(x)$

STEPS:

1. Pick first part of q by making highest power
2. Fill in rest of column
3. Pick next part of q to balance next highest power
4. Repeat until q is a constant term

If you divide cubic $f(x)$ by $(ax + b)$ then:

If $(ax + b)$ IS A FACTOR then: $f(x) = (ax + b)(ax^2 + bx + c)$ where $ax^2 + bx + c$ is the quotient from above

(you don't have to know this but: IF IT IS NOT A FACTOR then the remainder is found by calculating $f(-\frac{b}{a})$

Notes

Worked Example

497a: Determine a factor from algebraic division of a quadratic or cubic expression, given another factor (no zero coefficients).

Given that

$$\frac{3x^3 - 17x^2 + 8x + 48}{3x + 4} = Ax^2 + Bx + C$$

Use algebraic long division to work out the values of the constants A, B, and C.

$Ax^2 + Bx + C =$

① make table

$3x$		
$+4$		

divisor here

② reverse fill grid from top left

x^2 because $(3x) \times (x^2) = \underline{\underline{3x^3}}$

	x^2
$3x$	$3x^3$
$+4$	

③
- Fill in remainder of 1st column
- balance "x^2"

need $+4x^2 + \boxed{-21x^2} = -17x^2$

- Fill in remainder of column

	x^2	$-7x$
$3x$	$3x^3$	$-21x^2$
$+4$	$+4x^2$	$-28x$

④
- continue process

	x^2	$-7x$	$+12$
$3x$	$3x^3$	$-21x^2$	$+36x$
$+4$	$+4x^2$	$-28x$	$+48$

IF YOU ARE CORRECT BOTTOM LEFT is +48

⑤ State solution clearly

$$\frac{3x^3 - 17x^2 + 8x + 48}{3x + 4} = x^2 - 7x + 12$$

Worked Example

$f(x) = 4x^4 - 17x^2 + 4$

Divide $f(x)$ by $(2x + 1)$, giving your answer in the form $f(x) = (2x + 1)(ax^3 + bx^2 + cx + d)$.

The Factor Theorem

The Factor Theorem states that if $f(x)$ is a polynomial then:

- If $f\left(\frac{b}{a}\right) = 0$, then $(ax - b)$ is a factor of $f(x)$.
- Conversely, if $(ax - b)$ is a factor of $f(x)$, then $f\left(\frac{b}{a}\right) = 0$.

- When showing something is a factor in exam questions state 'by factor theorem'

- Model solution should finish with a sentence such as:

$f(-2) = 0$, by factor theorem $(x + 2)$ is a factor of $f(x)$

- If you are asked to use factor theorem you must attempt to find $f(a)$. You <u>must not</u> attempt the division of $f(x)$ by $(x - a)$

Notes

Worked Example

$$f(x) = 4x^3 + 28x^2 + 63x + 45.$$

(a) Use the factor theorem to show that $(x+3)$ is a factor of $f(x)$

(b) Hence fully factorise $f(x)$

Exam Q

P1 2022

2. $$f(x) = (x - 4)(x^2 - 3x + k) - 42 \quad \text{where } k \text{ is a constant}$$

Given that $(x + 2)$ is a factor of $f(x)$, find the value of k.

(3)

Your Turn

2. $$\mathrm{f}(x) = (x-2)(x^3 + 8x^2 - 20x + k) - 1225 \text{ where } k \text{ is a constant}$$

Given that $(x + 3)$ is a factor of $\mathrm{f}(x)$, find the value of k.

(3)

(Total for Question 2 is 3 marks)

Exam Q

P1 2019

1. $$f(x) = 3x^3 + 2ax^2 - 4x + 5a$$

Given that $(x + 3)$ is a factor of $f(x)$, find the value of the constant a.

(3)

Your Turn

1. $$f(x) = x^3 + 2bx^2 - 6x - b$$

Given that $(x + 1)$ is a factor of $f(x)$, find the value of the constant b.

(Total for Question 1 is 3 marks)

Past Paper Questions

A2 2021 Paper 1

Algebraic Methods

1.

$$f(x) = ax^3 + 10x^2 - 3ax - 4$$

Given that $(x - 1)$ is a factor of $f(x)$, find the value of the constant a.

You must make your method clear.

(3)

Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

A2 2021 Paper 1

Algebraic Methods

Question	Scheme	Marks	AOs
1	$f(1) = a(1)^3 + 10(1)^2 - 3a(1) - 4 = 0$	M1	3.1a
	$6 - 2a = 0 \Rightarrow a = ...$	M1	1.1b
	$a = 3$	A1	1.1b
		(3)	
			(3 marks)

Notes

Main method seen:
M1: Attempts $f(1) = 0$ to set up an equation in a It is implied by $a + 10 - 3a - 4 = 0$
Condone a slip but attempting $f(-1) = 0$ is M0
M1: Solves a linear equation in a .
Using the main method it is dependent upon having set $f(\pm 1) = 0$
It is implied by a solution of $\pm a \pm 10 \pm 3a \pm 4 = 0$.
Don't be concerned about the mechanics of the solution.
A1: $a = 3$ (following correct work)

Summary of Key Points

1 When simplifying an algebraic fraction, factorise the numerator and denominator where possible and then cancel common factors.

2 You can use long division to divide a polynomial by $(x \pm p)$, where p is a constant.

3 The **factor theorem** states that if $f(x)$ is a polynomial then:

- If $f(p) = 0$, then $(x - p)$ is a factor of $f(x)$
- If $(x - p)$ is a factor of $f(x)$, then $f(p) = 0$