KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 13
Applied Mathematics
M2 6 Projectiles

HGS Maths

Dr Frost Course

Name: ______________________________________

Class: ______________________________________

Contents

Prior knowledge check

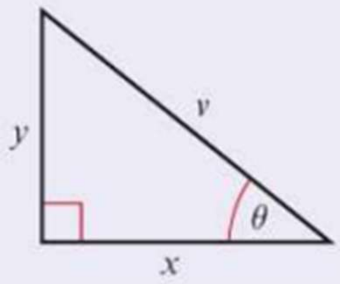
Prior knowledge check

1 A small ball is projected vertically upwards from a point P with speed 15 m s^{-1}. The ball is modelled as a particle moving freely under gravity. Find:

a the maximum height of the ball

b the time taken for the ball to return to P. ← **Year 1, Chapter 9**

2 Write expressions for x and y in terms of v and θ. ← **GCSE Mathematics**



3 **a** Given $\sin\theta = \frac{5}{13}$ find

i $\cos\theta$ **ii** $\tan\theta$

b Given $\tan\theta = \frac{8}{15}$ find

i $\sin\theta$ **ii** $\cos\theta$ ← **Pure Year 1, Chapter 10**

6.1) Horizontal projection

In **vertical** direction, acceleration downwards is g ms^{-2}.

Use suvat equations as before.

In **horizontal** direction, acceleration is 0 ms^{-2}.

Constant velocity, so can use bog standard $speed = \frac{distance}{time}$

- **The horizontal motion of a projectile is modelled as having constant velocity ($a = 0$). You can use the formula $s = vt$.**

The force due to gravity is modelled as being constant, so the vertical acceleration is constant.

- **The vertical motion of a projectile is modelled as having constant acceleration due to gravity ($a = g$).**

Use $g = 9.8$ m s^{-2} unless the question specifies a different value.

Links You can use the constant acceleration formulae for the vertical motion of a projectile:

$v = u + at$

$s = \left(\frac{u+v}{2}\right)t$

$v^2 = u^2 + 2as$

$s = ut + \frac{1}{2}at^2$

$s = vt - \frac{1}{2}at^2$

← Year 1, Chapter 9

654a: Determine the horizontal displacement of a particle projected horizontally.

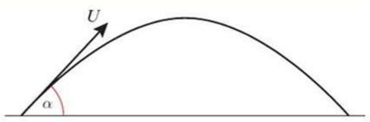
A ball is projected horizontally at 10.8 m s^{-1} from a point h m above the ground.

The ball hits the ground after 3.8 seconds.

Find the horizontal distance travelled by the ball before it reaches the ground.
Give your answer correct to 2 significant figures.

6.2) Horizontal and vertical components

Suppose a particle is projected with initial velocity U, at an angle α above the horizontal. The angle α is called the **angle of projection**.

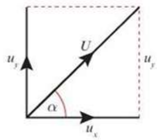


You can **resolve** the velocity into **components** that act horizontally and vertically:

Links This is the same technique as you use to resolve forces into components. ← Section 5.1

$\cos\alpha = \frac{u_x}{U}$ so $u_x = U\cos\alpha$

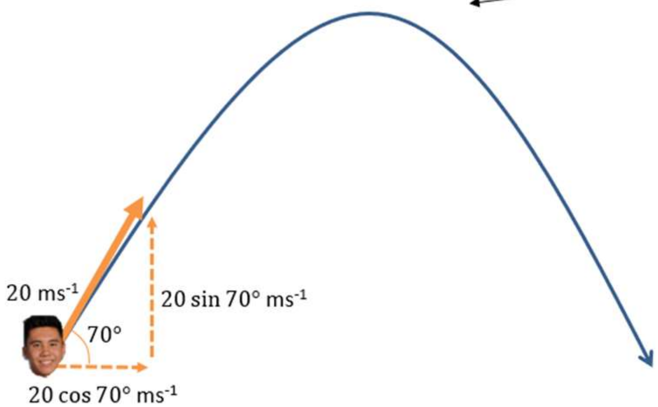
$\sin\alpha = \frac{u_y}{U}$ so $u_y = U\sin\alpha$



- **When a particle is projected with initial velocity U, at an angle α above the horizontal:**
  - **The horizontal component of the initial velocity is $U\cos\alpha$**
  - **The vertical component of the initial velocity is $U\sin\alpha$**

Just as **we split forces into its horizontal and vertical components**, in order to consider forces in the horizontal and vertical directions respectively, we can do **exactly the same with velocity**!

When the object is at its highest point:
The vertical velocity is 0.



We know that the scalar form of velocity is **speed**, and thus we just find the **magnitude** of the velocity vector:

$$\begin{pmatrix}12\\5\end{pmatrix} ms^{-1}$$

$$\Rightarrow \sqrt{12^2+5^2}$$
$$= 13\ ms^{-1}$$

Notes

Worked Example

A particle is projected from a point on a horizontal plane with an initial velocity of $39\ ms^{-1}$ at an angle α above the horizontal, where $\tan\alpha = \frac{5}{12}$.

a) Find the horizontal and vertical components of the initial velocity

b) Express the initial velocity as a vector in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$

T.112 : 6B Qs 1-4, P. 50 : 6.2 Qs 1,2,4,5,6

Worked Example

A particle is projected with velocity $\boldsymbol{U} = (2\boldsymbol{i} + 7\boldsymbol{j})\ ms^{-1}$ where $\boldsymbol{i}$ and $\boldsymbol{j}$ are the unit vectors in the horizontal and vertical directions respectively.
Find the initial speed of the particle and its angle of projection.

Worked Example

A particle is projected with velocity $\boldsymbol{U} = (5k\boldsymbol{i} + 2k\boldsymbol{j})\ ms^{-1}$.

a) Find the angle of projection

Given that the initial speed is $5\sqrt{29}\ ms^{-1}$

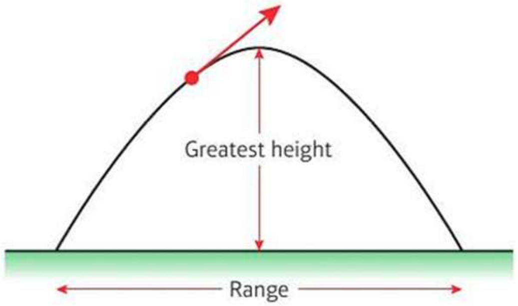
b) Find the possible values of k

T.112 : 6B Qs 5+, P. 50 : 6.2 Qs 3,7

6.3) Projection at any angle

You can solve problems involving particles projected at any angle by resolving the initial velocity into horizontal and vertical components.

The distance from the point from which the particle was projected to the point where it strikes the horizontal plane is called the **range**.



The time the particle takes to move from its point of projection to the point where it strikes the horizontal plane is called the **time of flight** of the projectile.

- **A projectile reaches its point of greatest height when the vertical component of its velocity is equal to 0.**

Worked Example

A ball is struck by a racket at a point which is $2\ m$ above horizontal ground. Immediately after the ball is truck, the ball has velocity $(10\boldsymbol{i} + 16\boldsymbol{j})ms^{-1}$ where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors horizontally and vertically respectively.
After being struck, the ball travels freely under gravity until it strikes the ground.
Find:

a) The greatest height above the ground reached by the ball
b) The speed of the ball as it reaches the ground
c) The angle the velocity of the ball makes with the ground as the ball reaches B

Worked Example

A particle is projected from a point on level ground with speed $U\ ms^{-1}$ and an angle of elevation of α.
The maximum height reached by the particle is $15.3061\ m$ (4 dp) above the ground and the particle hits the ground $35.3480\ m$ (4 dp) from its point of projection.
Find the value of α and U

T.117 : 6C Qs 6+, P. 51 : 6.3 Qs 4+

5.

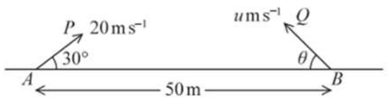


Figure 3

The points A and B lie 50 m apart on horizontal ground.

At time $t = 0$ two small balls, P and Q, are projected in the vertical plane containing AB.

Ball P is projected from A with speed $20\,\mathrm{m\,s^{-1}}$ at 30° to AB.

Ball Q is projected from B with speed $u\,\mathrm{m\,s^{-1}}$ at angle θ to BA, as shown in Figure 3.

At time $t = 2$ seconds, P and Q collide.

Until they collide, the balls are modelled as particles moving freely under gravity.

(a) Find the velocity of P at the instant before it collides with Q.

(6)

(b) Find

(i) the size of angle θ,

(ii) the value of u.

(6)

(c) State one limitation of the model, other than air resistance, that could affect the accuracy of your answers.

(1)

5.

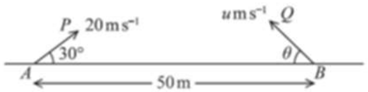


Figure 3

The points A and B lie 50 m apart on horizontal ground.

At time $t = 0$ two small balls, P and Q, are projected in the vertical plane containing AB.

Ball P is projected from A with speed 20 m s^{-1} at 30° to AB.

Ball Q is projected from B with speed u m s^{-1} at angle θ to BA, as shown in Figure 3.

At time $t = 1$ second, P and Q collide.

Until they collide, the balls are modelled as particles moving freely under gravity.

(*a*) Find the magnitude and the direction of the velocity of P at the instant before it collides with Q.
(6)

(*b*) Find

(i) the size of angle θ,

(ii) the value of u.
(6)

(*c*) State one limitation of the model, other than air resistance, that could affect the accuracy of your answers.
(1)

(Total for Question 5 is 13 marks)

5.

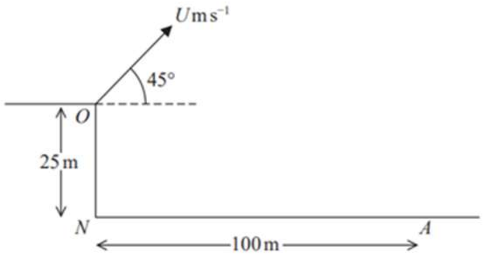


Figure 2

A small ball is projected with speed U m s^{-1} from a point O at the top of a vertical cliff.

The point O is 25 m vertically above the point N which is on horizontal ground.

The ball is projected at an angle of 45° above the horizontal.

The ball hits the ground at a point A, where $AN = 100$ m, as shown in Figure 2.

The motion of the ball is modelled as that of a particle moving freely under gravity.

Using this initial model,

(a) show that $U = 28$

(6)

(b) find the greatest height of the ball above the horizontal ground NA.

(3)

In a refinement to the model of the motion of the ball from O to A, the effect of air resistance is included.

This refined model is used to find a new value of U.

(c) How would this new value of U compare with 28, the value given in part (a)?

(1)

5

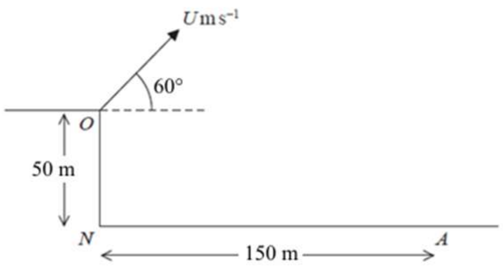


Figure 2

A small ball is projected with speed U m s^{-1} from a point O at the top of a vertical cliff.

The point O is 50 m vertically above the point N which is on horizontal ground.

The ball is projected at an angle of 60° above the horizontal.

The ball hits the ground at a point A, where $AN = 150$ m, as shown in Figure 2.

The motion of the ball is modelled as that of a particle moving freely under gravity.

Using this initial model,

(*a*) show that $U = 37.7$ to 3 significant figures

(6)

(*b*) taking $U = 37.7$, find the greatest height of the ball above the horizontal ground NA.

(3)

In a refinement to the model of the motion of the ball from O to A, the effect of air resistance is included.

This refined model is used to find a new value of U.

(*c*) How would this new value of U compare with 37.7, the value given in part (a)?

(1)

(*d*) State one further refinement to the model that would make the model more realistic.

(1)

(Total for Question 5 is 11 marks)

6.4) Projectile motion formulae

Exam Note: You may be asked to derive these. But don't attempt to memorise them or actually use them to solve exam problems – instead use the techniques used earlier in the chapter.

For a particle projected with initial velocity U at angle α above horizontal and moving freely under gravity:

- Time of flight $= \frac{2U \text{ si}}{g}$
- Time to reach greatest height $= \frac{U \sin \alpha}{g}$
- Range on horizontal plane $= \frac{U^2 \sin 2\alpha}{g}$
- Equation of trajectory: $y = x \tan \alpha - \frac{gx^2}{2U^2}(1 + \tan^2 \alpha)$
  where y is vertical height of particle and x horizontal distance.

Worked Example

A particle is projected from a point with speed U at an angle of elevation α and moves freely under gravity. When the particle has moved a horizontal distance x, its height above the point of projection is y.

(a) Show that $y = x \tan\alpha - \frac{gx^2}{2u^2}(1 + \tan^2\alpha)$

A particle is projected from a point O on a horizontal plane, with speed $14\ ms^{-1}$ at an angle of elevation α. The particle passes through a point B, which is at a horizontal distance of 16m from O and at a height of 4m above the plane.

(b) Find the two possible values of α, giving your answers to the nearest degree.

T.123 : 6D Qs all, P. 52 : 6.4 Qs all

A small ball is projected with speed $28\,\mathrm{m\,s^{-1}}$ from a point O on horizontal ground.

After moving for T seconds, the ball passes through the point A.

The point A is 40 m horizontally and 20 m vertically from the point O, as shown in Figure 2.

The motion of the ball from O to A is modelled as that of a particle moving freely under gravity.

Given that the ball is projected at an angle α to the ground, use the model to

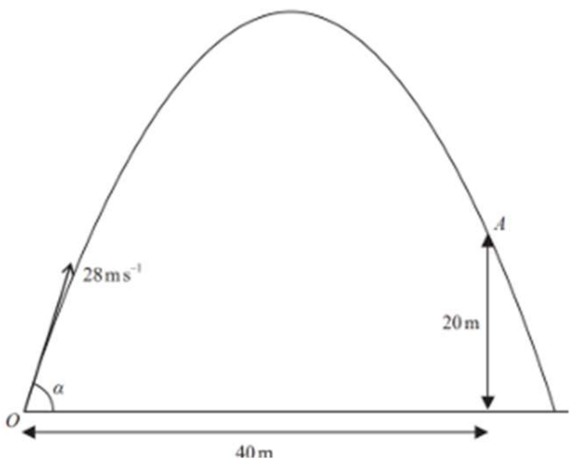
(a) show that $T = \dfrac{10}{7\cos\alpha}$ **(2)**

(b) show that $\tan^2\alpha - 4\tan\alpha + 3 = 0$ **(5)**

(c) find the greatest possible height, in metres, of the ball above the ground as the ball moves from O to A. **(3)**

The model does not include air resistance.

(d) State one other limitation of the model. **(1)**



Your Turn

5.

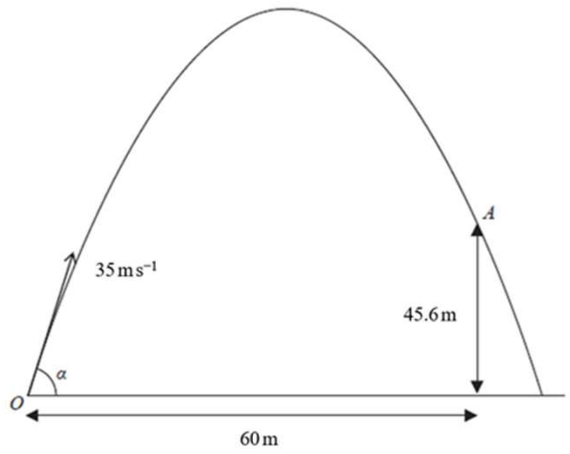


Figure 2

A small ball is projected with speed $35\,\text{m}\,\text{s}^{-1}$ from a point O on horizontal ground.

After moving for T seconds, the ball passes through the point A.

The point A is 60 m horizontally and 45.6 m vertically from the point O, as shown in Figure 2.

The motion of the ball from O to A is modelled as that of a particle moving freely under gravity.

Given that the ball is projected at an angle α to the ground, use the model to

(a) show that $T = \dfrac{12}{7\cos\alpha}$

(2)

(b) show that $6\tan^2\alpha - 25\tan\alpha + 25 = 0$

(5)

(c) find the greatest possible height, in metres, of the ball above the ground as the ball moves from O to A.

(3)

The model does not include air resistance.

(d) State one other limitation of the model.

(1)

(Total for Question 5 is 11 marks)

At time $t = 0$, a small stone is projected with velocity $35\,\text{m s}^{-1}$ from a point O on horizontal ground.

The stone is projected at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$

In an initial model

- the stone is modelled as a particle P moving freely under gravity
- the stone hits the ground at the point A

Figure 4 shows the path of P from O to A.

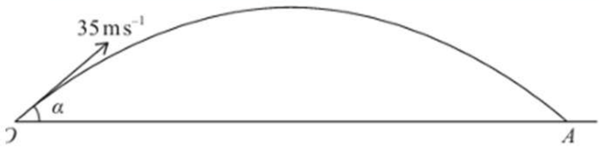


Figure 4

For the motion of P from O to A

- at time t seconds, the horizontal distance of P from O is x metres
- at time t seconds, the vertical distance of P above the ground is y metres

(a) Using the model, show that

$$y = \frac{3}{4}x - \frac{1}{160}x^2$$

(6)

(b) Use the answer to (a), or otherwise, to find the length OA.

(2)

Using the model, the greatest height of the stone above the ground is found to be H metres.

(c) Use the answer to (a), or otherwise, to find the value of H.

(2)

- The model is refined to include air resistance.

Using this new model, the greatest height of the stone above the ground is found to be K metres.

(d) State which is greater, H or K, justifying your answer.

(1)

(e) State one limitation of this refined model.

(1)

5.

Figure 4

At time $t = 0$, a small stone is projected with velocity $91\,\text{m s}^{-1}$ from a point O on horizontal ground.

The stone is projected at an angle α to the horizontal, where $\tan \alpha = \frac{5}{12}$

In an initial model

- the stone is modelled as a particle P moving freely under gravity
- the stone hits the ground at the point A

Figure 4 shows the path of P from O to A.

For the motion of P from O to A

- at time t seconds, the horizontal distance of P from O is x metres
- at time t seconds, the vertical distance of P above the ground is y metres

(*a*) Using the model, show that

$$y = \frac{5}{12}x - \frac{1}{1440}x^2$$

(6)

(*b*) Use the answer to (*a*), or otherwise, to find the length OA.

(2)

Using the model, the greatest height of the stone above the ground is found to be H metres.

(*c*) Use the answer to (*a*), or otherwise, to find the value of H.

(2)

- The model is refined to include air resistance.

Using this new model, the greatest height of the stone above the ground is found to be K metres.

(*d*) State which is greater, H or K, justifying your answer.

(1)

(*e*) State one limitation of this refined model.

(1)

(Total for Question 5 is 12 marks)

10.

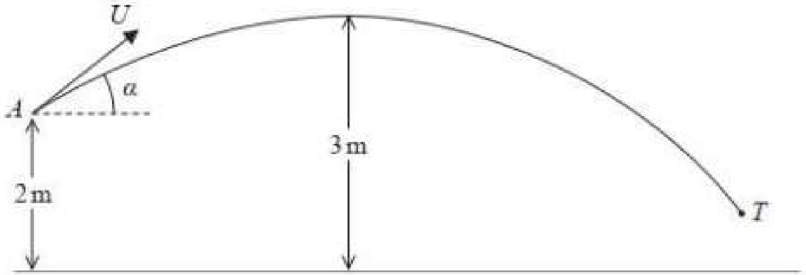


Figure 4

A boy throws a ball at a target. At the instant when the ball leaves the boy's hand at the point A, the ball is 2 m above horizontal ground and is moving with speed U at an angle α above the horizontal.

In the subsequent motion, the highest point reached by the ball is 3 m above the ground. The target is modelled as being the point T, as shown in Figure 4.
The ball is modelled as a particle moving freely under gravity.

Using the model,

(a) show that $U^2 = \dfrac{2g}{\sin^2\alpha}$.

(2)

The point T is at a horizontal distance of 20 m from A and is at a height of 0.75 m above the ground. The ball reaches T without hitting the ground.

(b) Find the size of the angle α

(9)

(c) State one limitation of the model that could affect your answer to part (b).

(1)

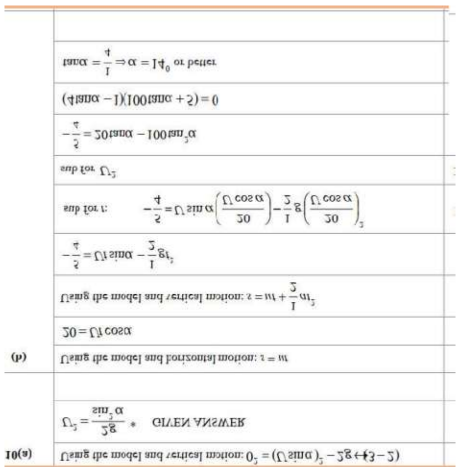
(d) Find the time taken for the ball to travel from A to T.

(3)

Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com



Summary of Key Points

Summary of key points

1 The **horizontal** motion of a projectile is modelled as having **constant velocity** ($a = 0$).
You can use the formula $s = vt$.

2 The **vertical** motion of a projectile is modelled as having **constant acceleration** due to gravity ($a = g$).

3 When a particle is projected with initial velocity U, at an angle α above the horizontal:
- The **horizontal component** of the initial velocity is $U \cos \alpha$
- The **vertical component** of the initial velocity is $U \sin \alpha$

4 A projectile reaches its point of greatest height when the vertical component of its velocity is equal to 0.

5 For a particle which is projected from a point on a horizontal plane with an initial velocity U at an angle α above the horizontal, and that moves freely under gravity:
- Time of flight $= \dfrac{2U \sin \alpha}{g}$
- Time to reach greatest height $= \dfrac{U \sin \alpha}{g}$
- Range on horizontal plane $= \dfrac{U^2 \sin 2\alpha}{g}$
- Equation of trajectory: $y = x \tan \alpha - gx^2 \dfrac{(1 + \tan^2 \alpha)}{2U^2}$

where y is the vertical height of the particle, x is the horizontal distance from the point of projection, and g is the acceleration due to gravity.

T.125 : mixed ex, P. 54 : BSG