KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 13
Pure Mathematics
9 Differentiation
(part 1)

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

Prior knowledge check

Prior knowledge check

1 Differentiate:

a $3x^2 - 5x$ **b** $\frac{2}{x} - \sqrt{x}$

c $4x^2(1 - x^2)$ ← **Year 1, Chapter 12**

2 Find the equation of the tangent to the curve with equation $y = 8 - x^2$ at the point $(3, -1)$. ← **Year 1, Chapter 12**

3 The curve C is defined by the parametric equations

$$x = 3t^2 - 5t, \quad y = t^3 + 2, \quad t \in \mathbb{R}$$

Find the coordinates of any points where C intersects the coordinate axes.

← **Section 8.4**

4 Solve $2 \operatorname{cosec} x - 3 \sec x = 0$ in the interval $0 \leqslant x \leqslant 2\pi$, giving your answers correct to 3 significant figures. ← **Section 6.3**

9.1) Differentiating $\sin x$ and $\cos x$

You need to know:

$y =$ *or* $f(x)=$	$\frac{dy}{dx} =$ *or* $f'(x) =$
ax^n	nax^{n-1}
e^{kx}	ke^{kx}
$\sin(kx)$	$k\cos(kx)$
$\cos(kx)$	$-k\sin(kx)$

You also are required by the exam specification to be familiar with deriving the $\sin(kx)$ and $\cos(kx)$ derivatives from first principles (on next page)

Notes

Worked Example

A curve has equation $y = \frac{1}{4}x - \cos 3x$.

Find the stationary points on the curve in the interval $0 \leq x \leq \pi$

Worked Example

A curve has equation $y = \sin 5x + 3x$. Find the stationary points on the curve in the interval $0 \leq x \leq \frac{3}{5}\pi$

T.234: Q 4-6 ,P.70 Q 2,3

Worked Example

A curve has equation $y = \sin 4x - \cos 3x$. Find the equation of the tangent to the curve at the point $(\pi, 1)$

Worked Example

A curve has equation $y = 3x^2 - \sin x$. Show that the equation of the normal to the curve at the point with x-coordinate π is

$$x + (6\pi + 1)y - \pi(18\pi^2 - 3\pi - 1) = 0$$

T.234: 7+,P.70 Q 4+

Proof for derivative of $sinx$

Things of helpfulness:

- As $x \to 0$, $\sin x \approx x$ and $\cos x \approx 1 - \frac{1}{2}x^2$
- $\sin(a+b) =$ $\sin a \cos b + \cos a \sin b$

If $y = f(x)$ then $\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

Use addition formula.

$$= \lim_{h\to 0} \left(\left(\frac{\cos h - 1}{h} \right) \sin x + \left(\frac{\sin h}{h} \right) \cos x \right)$$

Collect $\sin x$ terms and $\cos x$ terms together.

Since $\lim_{h\to 0} \sin h \approx h$, $\therefore \lim_{h\to 0} \frac{\sin h}{h} \approx \lim_{h\to 0} \frac{h}{h} = 1$

Similarly, $\lim_{h\to 0} \frac{\cos h - 1}{h} \approx \lim_{h\to 0} \frac{1 - \frac{1}{2}h^2 - 1}{h} = \lim_{h\to 0} (-\frac{1}{2}h) \to 0$

Use small angle approximations to consider what the expressions in terms of h will become.

$$\therefore \frac{dy}{dx} = \lim_{h\to 0} (0 \sin x + 1 \cos x) = \cos x$$

YOUR TURN: Proof for derivative of $cosx$

Things of helpfulness:

- As $x \to 0$, $\sin x \approx x$ and $\cos x \approx 1 - \frac{1}{2}x^2$
- $\sin(a + b) =$ $\sin a \cos b + \cos a \sin b$

Use addition formula.

Collect $\sin x$ terms and $\cos x$ terms together.

Use small angle approximations to consider what the expressions in terms of h will become.

9.2) Differentiating exponentials and logarithms

LEARN:

$y =$ or $f(x) =$	$\frac{dy}{dx} =$ or $f'(x) =$
ax^n	nax^{n-1}
e^{kx}	ke^{kx}
$\ln(x)$	$\frac{1}{x}$
a^x	$a^x \ln(a)$

***these results can be proved later in the chapter.**

Worked Example

Find:

$\frac{d}{dx}(\ln x^2)$ $\frac{d}{dx}(\ln x^3)$ $\frac{d}{dx}(\ln x^4)$

Worked Example

Find:

$$\frac{d}{dx}\left(\frac{3+2e^{5x}}{7e^{4x}}\right)$$

$$\frac{d}{dx}\left(\frac{7-5e^{-4x}}{3e^{2x}}\right)$$

Worked Example

Find:

$$\frac{d}{dx}(e^x - 3)^2$$

T.236: Q1-3,P.70 Q 1-3

Worked Example

The population P of a species after t days can be modelled using $P = 640(2^{-3t})$

a) Determine how many days have elapsed before there are 20 individuals left
b) Determine the rate of change of the population after 3 days.

Worked Example

A curve has the equation

$$y = e^{4x} - \ln x$$

Show that the equation of the tangent to the curve at the point with x-coordinate 1 is: $y = (4e^4 - 1)x - 3e^4 + 1$

Worked Example

A curve has the equation

$$y = 5(2^{3x})$$

Find the equation of the normal to the curve at the point with x-coordinate 1 in the form $y = ax + b$, where a and b are constants to be found in exact form

T.236: Q4+,P.70 Q 4+

9.3) The chain rule

The chain rule allows us to differentiate a composite function, i.e. a function within a function.

Example: differentiate $y = (3x^4 + x)^5$

The Chain Rule:
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Fro Note: Notice how the du's sort of 'cancel' top and bottom on the RHS. This is not a valid proof of the chain rule, but the d's sort of behave as quantities which can often be manipulated in this way.

Full Method:

Let $u = 3x^4 + x$

In the Chain Rule, u represents the inner function.

$$\frac{du}{dx} = 12x^3 + 1$$

$$y = u^5$$

This represents the 'outer' function.

$$\frac{dy}{du} = 5u^4 = 5(3x^4 + x)^4$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 5(3x^4 + x)^4(12x^3 + 1)$$

Doing it mentally in one go:
(aka the 'bla method')

The chain rule boils down to:
$\frac{dy}{dx}$ = outer function differentiated
× inner function differentiated

$$\frac{dy}{dx} = 5(3x^4 + x)^4 \times (12x^3 + 1)$$

When differentiating the outer function, replace (in your head) the inner function with the word 'bla'. i.e. "What does bla^5 differentiate to?". It's 5 bla^4

Now differentiate the inner function, i.e. the 'bla'.

Worked Example

Differentiate with respect to x:

$y = e^{x^2+x}$

$f(x) = e^{x^3-2x+1}$

Worked Example

Differentiate with respect to x:

$$y = \ln(\sin x)$$

T.239: Q1,2,P.71 Q 1-3

Worked Example

Differentiate with respect to x:

$y = \sin^4 x$ $\qquad$ $f(x) = \cos^3 x$

Worked Example

Differentiate with respect to x:

$$y = e^{e^{-x}}$$

Worked Example

A curve C has equation

$$y = \frac{3}{(2-5x)^4}, x \neq \frac{2}{5}$$

Find an equation for the normal to C at the point with x-coordinate 1 in the form $ax + by + c = 0$, where a, b and c are integers

T.239: Q3-5,P.71 Q 4-8

The reciprocal case

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

Find $\frac{dy}{dx}$ when $x = 2y^2 + y$

$$\boldsymbol{\frac{dx}{dy} = 4y + 1}$$

$$\boldsymbol{\frac{dy}{dx} = \frac{1}{4y + 1}}$$

Find the gradient of $x = (1 + 2y)^3$ when $y = 1$

$$\boldsymbol{\frac{dx}{dy} = 6(1 + 2y)^2} \qquad \rightarrow \quad \boldsymbol{\frac{dy}{dx} = \frac{1}{6(1 + 2y)^2}}$$

$$\boldsymbol{m = \frac{1}{6(1 + 2)^2} = \frac{1}{54}}$$

Worked Example

Find the gradient of:

$x = (1 + 2y)^3$ when $y = 1$

$x = (3y^2 - 2)^4$ at $(1, -1)$

Worked Example

Find $\frac{dy}{dx}$ when:

$x = 2y^2 + y$

$x = 3y^4 - y^2 - 1$

T.239: Q6+

14. The curve C, in the standard Cartesian plane, is defined by the equation

$$x = 4\sin 2y \qquad \frac{-\pi}{4} < y < \frac{\pi}{4}$$

The curve C passes through the origin O

(a) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the origin.

(2)

(b) (i) Use the small angle approximation for $\sin 2y$ to find an equation linking x and y for points close to the origin.

(ii) Explain the relationship between the answers to (a) and (b)(i).

(2)

(c) Show that, for all points (x, y) lying on C,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{a\sqrt{b - x^2}}$$

where a and b are constants to be found.

(3)

Your Turn

14. The curve C, in the standard Cartesian plane, is defined by the equation

$$x = 9 \sin 2y \qquad -\frac{\pi}{4} < y < \frac{\pi}{4}$$

The curve C passes through the origin O

(a) Find the value of $\frac{dy}{dx}$ at the origin.

(2)

(b) (i) Use the small angle approximation for $\sin 2y$ to find an equation linking x and y for points close to the origin.

(ii) Explain the relationship between the answers to part (a) and part (b)(i).

(2)

(c) Show that, for all points (x, y) lying on C,

$$\frac{dy}{dx} = \frac{1}{a\sqrt{b - x^2}}$$

where a and b are constants to be found.

(3)

(Total for Question 14 is 7 marks)

11.

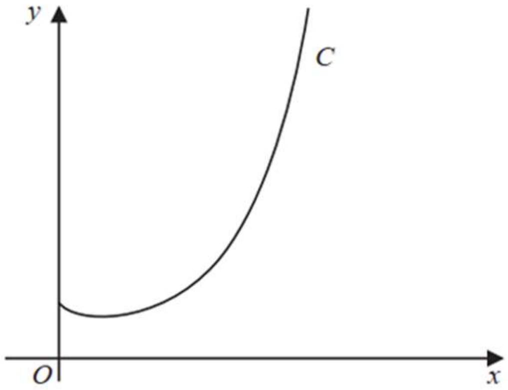

Figure 8

Figure 8 shows a sketch of the curve C with equation $y = x^x$, $x > 0$

(a) Find, by firstly taking logarithms, the x coordinate of the turning point of C.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(5)

Your Turn

1.

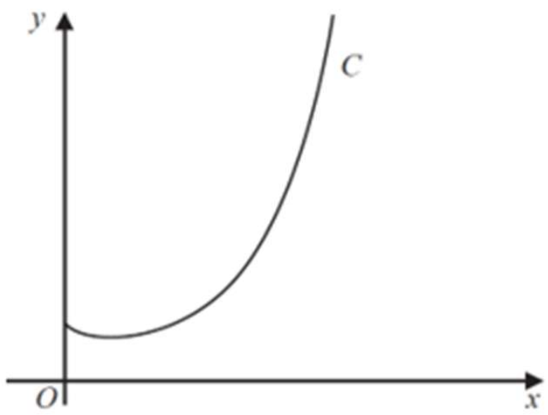

Figure 8

Figure 8 shows a sketch of the curve C with equation

$$y = x^x, \quad x > 0.$$

(a) Find, by firstly taking logarithms, the x coordinate of the turning point of C.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(5)

9.4) The product rule

As mentioned previously, the product rule is used, unsurprisingly, when we have a **product** of two functions.

The product rule:
If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Or simply: $y' = uv' + vu'$

If $y = x^2 \sin x$, determine $\frac{dy}{dx}$

$u = x^2 \qquad v = \sin x$

Write out $u = \cdots, v = \cdots$ then differentiate each.

$u' = 2x \qquad v' \cos x$

$$\frac{dy}{dx} = x^2 \cos x + 2x \sin x$$

Sub into product rule.

If $y = xe^{2x}$, determine the coordinates of the turning point.

$u = x \qquad v = e^{2x}$

$u' = 1 \qquad v' = 2e^{2x}$

$$\frac{dy}{dx} = 2xe^{2x} + e^{2x} = 0$$

Notes

Worked Example

Differentiate with respect to x:

$y = e^{3x}\sin^4 2x$ $\qquad\qquad$ $f(x) = e^{2x}\cos^3 4x$

Worked Example

Determine the coordinates of the turning point:

$y = xe^{3x}$ $\quad\quad\quad\quad$ $f(x) = xe^{4x}$

T.242: Q1-3,P.72 Q 1-5

Worked Example

Differentiate with respect to x:

$$y = x^2\sqrt{5x - 2}$$

Worked Example

Find the equation of the tangent to the curve with equation $y = x^2 \sin(x^2)$ at the point $\left(\frac{\sqrt{\pi}}{2}, \frac{\pi\sqrt{2}}{8}\right)$ in the form $ax + by + c = 0$ where a, b and c are exact constants.

Worked Example

Differentiate with respect to x:

$$\frac{dy}{dx} = \frac{1}{5x(x+1)^{\frac{1}{2}}}$$

T.242: Q4+,P.72 Q 6+

12. $$f(x) = 10e^{-0.25x} \sin x, \quad x \geqslant 0$$

(a) Show that the x coordinates of the turning points of the curve with equation $y = f(x)$ satisfy the equation $\tan x = 4$

(4)

Figure 3 shows a sketch of part of the curve with equation $y = f(x)$.

(b) Sketch the graph of H against t where

$$H(t) = \left|10e^{-0.25t} \sin t\right| \quad t \geqslant 0$$

showing the long-term behaviour of this curve.

(2)

The function $H(t)$ is used to model the height, in metres, of a ball above the ground t seconds after it has been kicked.

Using this model, find

(c) the maximum height of the ball above the ground between the first and second bounce.

(3)

(d) Explain why this model should not be used to predict the time of each bounce.

(1)

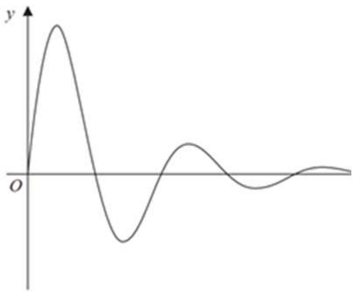

Figure 3

Your Turn

12. $$\mathrm{f}(x) = 12\mathrm{e}^{-\frac{x}{3}} \sin x, \qquad x \geq 0$$

(*a*) Show that the x coordinates of the turning points of the curve with equation $y = \mathrm{f}(x)$ satisfy the equation $\tan x = 3$

(4)

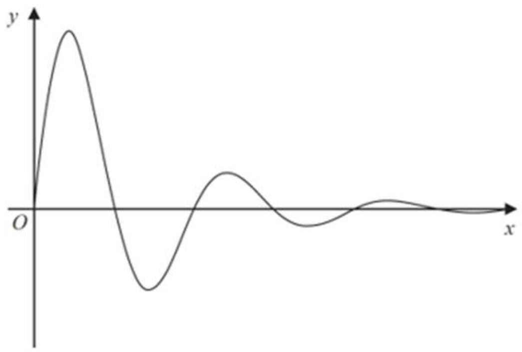

Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$.

(*b*) Sketch the graph of H against t where

$$\mathrm{H}(t) = |12\mathrm{e}^{-\frac{x}{3}} \sin t| \qquad t \geq 0$$

showing the long-term behaviour of this curve.

(2)

The function $\mathrm{H}(t)$ is used to model the height, in metres, of a ball above the ground t seconds after it has been kicked.

Using this model, find

(*c*) the maximum height of the ball above the ground between the first and second bounce.

(3)

(*d*) Explain why this model should not be used to predict the time of each bounce.

(1)

(Total for Question 12 is 10 marks)

Figure 2 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 4(x^2 - 2)\mathrm{e}^{-2x} \qquad x \in \mathbb{R}$$

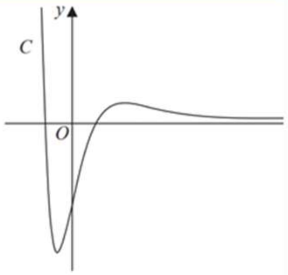

Figure 2

(a) Show that $\mathrm{f}'(x) = 8(2 + x - x^2)\mathrm{e}^{-2x}$

(3)

(b) Hence find, in simplest form, the exact coordinates of the stationary points of C.

(3)

The function g and the function h are defined by

$$\mathrm{g}(x) = 2\mathrm{f}(x) \qquad x \in \mathbb{R}$$

$$\mathrm{h}(x) = 2\mathrm{f}(x) - 3 \qquad x \geqslant 0$$

(c) Find (i) the range of g

(ii) the range of h

(3)

Your Turn

9.

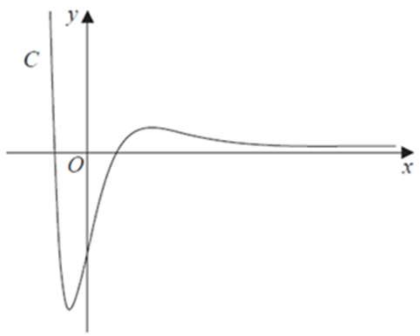

Figure 2

Figure 2 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 9(x^2 - 6)\mathrm{e}^{-2x} \qquad x \in \mathbb{R}$$

(*a*) Show that $\mathrm{f}'(x) = 18(6 + x - x^2)\mathrm{e}^{-2x}$

(3)

(*b*) Hence find, in simplest form, the exact coordinates of the stationary points of C.

(3)

The function g and the function h are defined by

$$\mathrm{g}(x) = 2\mathrm{f}(x) \qquad x \in \mathbb{R}$$

$$\mathrm{h}(x) = 2\mathrm{f}(x) - 2 \qquad x \geq 0$$

(*c*) Find (i) the range of g,

(ii) the range of h.

(3)

(Total for Question 9 is 9 marks)

The quotient rule:

If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

Worked Example

597h: **Use the quotient rule with simple functions.**

Given that

$$y = \frac{2\sin(x)}{\ln(x)}$$

find $\dfrac{dy}{dx}$.

Worked Example

Find the stationary point of

$$y = \frac{\cos x}{e^{3x}}, 0 < x < \pi$$

Worked Example

Find the equation of the tangent to the curve $y = \frac{e^{\frac{1}{4}x}}{x}$ at the point $(4, \frac{1}{4}e)$

T.245: Q5+,P.73 Q 4+

14. Given that

$$y = \frac{x-4}{2+\sqrt{x}} \qquad x > 0$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{A\sqrt{x}} \qquad x > 0$$

where A is a constant to be found.

(4)

Your Turn

14. Given that

$$y = \frac{x-16}{4+\sqrt{x}} \qquad x > 0$$

show that

$$\frac{dy}{dx} = \frac{1}{A\sqrt{x}} \qquad x > 0$$

where A is a constant to be found.

(4)

(Total for Question 14 is 4 marks)