KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 13
Pure Mathematics
9 Differentiation
(part 2)

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

9.6) Differentiating trigonometric functions

LEARN:

$y =$ or $f(x)=$	$\frac{dy}{dx} =$ or $f'(x) =$
ax^n	nax^{n-1}
e^{kx}	ke^{kx}
$\sin(kx)$	$k\cos(kx)$
$\cos(kx)$	$-k\sin(kx)$
$\ln(x)$	$\frac{1}{x}$
a^x	$a^x\ln(a)$

9.6) Differentiating trigonometric functions

You are given:

Differentiation

First Principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

f(x)	**f′(x)**
$\tan kx$	$k \sec^2 kx$
$\sec kx$	$k \sec kx \tan kx$
$\cot kx$	$-k \operatorname{cosec}^2 kx$
$\operatorname{cosec} kx$	$-k \operatorname{cosec} kx \cot kx$
$\dfrac{f(x)}{g(x)}$	$\dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

Worked Example

Differentiate with respect to x:

$$y = \tan^4 2x$$

$$y = \cot^4 2x$$

T.248: Q 1-3 ,P.74: Q1-2

Worked Example

Differentiate with respect to x:

$$y = \frac{cosec\ 3x}{x^3}$$

T.248: Q 4-7 ,P.74: Q3-7

Worked Example

Given that $x = \cot y$, express $\frac{dy}{dx}$ in terms of x.

Worked Example

Differentiate with respect to x:

$y = \arccos x$

$y = \arctan x$

Worked Example
Differentiate with respect to x: $y = \arccos x^4$ $\quad\quad\quad\quad\quad\quad\quad\quad$ $y = \arctan x^3$

Worked Example

Given that $x = \operatorname{cosec} 3y$, find $\frac{dy}{dx}$ in terms of x

Worked Example

Given that $y = \arctan\left(\frac{1+x}{1-x}\right)$, find $\frac{dy}{dx}$

T.248: Q 8+ ,P.74: Q8

9.7) Parametric differentiation

Recall from the previous chapter that parametric equations are when we define each of x and y (and possibly z) in terms of some separate parameter, e.g. t.

If x and y are given as functions of a parameter t, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{dy}{dt} \times \frac{dt}{dx}$$

**this will give you a function in terms of parameter t*

To find the gradient at a specific point you will need to substitute in a *t* value

Worked Example

Find the gradient at the point P where $t = 3$, on the curve given parametrically by

$$x = t^2 - t, \qquad y = \mathrm{t}^4 - 2, \qquad \mathrm{t} \in \mathbb{R}$$

T.251: Q 1 ,P.75: Q1

Worked Example

Find the equation of the tangent at the point where $t = \frac{\pi}{6}$, to the curve with parametric equations

$$x = \sqrt{5}\sin 2t\,, \qquad y = 8\cos^2 t\,, \qquad 0 \le t \le \pi$$

Worked Example

Find the equation of the normal at the point where $\theta = \frac{\pi}{3}$, to the curve with parametric equations

$$x = 2\cos\theta\,, \qquad y = 7\sin\theta$$

T.251: Q2+ ,P.75: Q2+

13. The curve C has parametric equations

$$x = \sin 2\theta \qquad y = \operatorname{cosec}^3 \theta \qquad 0 < \theta < \frac{\pi}{2}$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of θ

(3)

(b) Hence find the exact value of the gradient of the tangent to C at the point where $y = 8$

(3)

Your Turn

13. The curve C has parametric equations

$$x = \cos 2\theta \qquad y = \sec^3 \theta \qquad 0 < \theta < \frac{\pi}{2}$$

(*a*) Find an expression for $\frac{dy}{dx}$ in terms of θ

(3)

(*b*) Hence find the exact value of the gradient of the tangent to C at the point where $y = 8$

(3)

(Total for Question 13 is 6 marks)

Figure 6 shows a sketch of the curve C with parametric equations

$$x = 2\tan t + 1 \qquad y = 2\sec^2 t + 3 \qquad -\frac{\pi}{4} \leqslant t \leqslant \frac{\pi}{3}$$

The line l is the normal to C at the point P where $t = \frac{\pi}{4}$

(a) Using parametric differentiation, show that an equation for l is

$$y = -\frac{1}{2}x + \frac{17}{2}$$

(5)

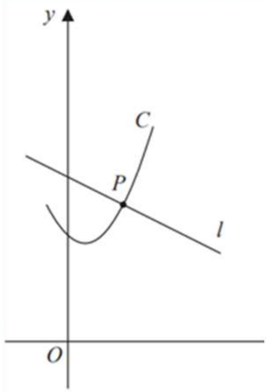


Figure 6

(b) Show that all points on C satisfy the equation

$$y = \frac{1}{2}(x-1)^2 + 5$$

(2)

The straight line with equation

$$y = -\frac{1}{2}x + k \qquad \text{where } k \text{ is a constant}$$

intersects C at two distinct points.

(c) Find the range of possible values for k.

Your Turn

16.

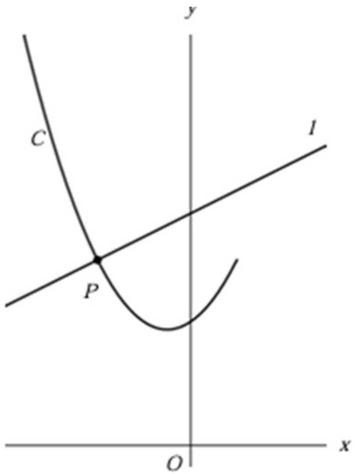


Figure 6

Figure 6 shows a sketch of the curve C with parametric equations

$$x = 3\cot t - 1 \qquad y = 3\,\mathrm{cosec}^2\, t + 2 \qquad \frac{\pi}{4} \leq t < \theta$$

The line l is the normal to C at the point P where $t = \frac{3\pi}{4}$

(*a*) Using parametric differentiation, show that an equation for l is

$$y = \frac{1}{2}x + 10$$

(5)

(*b*) Show that all points on C satisfy the equation

$$y = \frac{1}{3}(x+1)^2 + 5$$

(2)

The straight line with equation

$$y = \frac{1}{2}x + k \quad \text{where } k \text{ is a constant}$$

intersects C at two distinct points.

(*c*) Find the range of possible values for k.

(5)

(Total for Question 16 is 12 marks)

9.8) Implicit differentiation

This is used when a function is not written with y as the subject, e.g. $y = f(x)$ (i.e. y explicitly in terms of x)
For example, a circle equation : $x^2 + y^2 = 1$

Key things to know: when you differentiate implicitly:

1. All the differentiation rules still apply whether the variable is x or y
2. HOWEVER, if you differentiate y with respect to x, you must multiply the result

EXAMPLE:

$$x^2 + y^2 = 1$$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$$

Differentiate each term

$\frac{d}{dx}(x^2) = 2x$ $\quad$ $\frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx}$ $\quad$ $\frac{d}{dx}(1) = 0$

So, we get:

$$2x + 2y\frac{dy}{dx} = 0$$

Rearrange to isolate $\frac{dy}{dx}$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

The **differential of xy** is worth learning too!

Apply the product rule

$$\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$$

Notes

Worked Example

Find $\frac{dy}{dx}$ where:

$$x^4 - x + y^2 - 3y = 5$$

T.255: Q 1-3 ,P.76: Q1-3

Worked Example

Find $\frac{dy}{dx}$ at the point $(1, 1)$, given that:

$$6x^2y - \frac{4x}{y^2} = 2$$

Worked Example

A curve is described by:

$$x^3 + 4y^2 = -12xy$$

Find the gradient of the curve at the points where $x = 8$

Worked Example

$$x^2 + y^2 + 20x + 4y - 8xy = -75$$

Find the values of y for which $\frac{dy}{dx} = 0$

T.255: Q 4-6,P.76: Q4-7

Worked Example

A curve has equation

$$x^2 + 4xy + y^2 - x = 35$$

Find the equation of the tangent to the curve at the point $(2, 3)$.
Give your answer in the form $ax + by + c = 0$, where a, b and c are integers

Worked Example

The curve $ye^{-4x} = 4x - y^2$.
Find the equation of the normal(s) to the curve at the point where $x = 0$.
Give your answer in the form $ax + by + c = 0$

T.256: Q 7+6,P.76: Q8

15. The curve C has equation

$$x^2 \tan y = 9 \qquad 0 < y < \frac{\pi}{2}$$

(a) Show that

$$\frac{dy}{dx} = \frac{-18x}{x^4 + 81}$$

(4)

(b) Prove that C has a point of inflection at $x = \sqrt[4]{27}$

(3)

Your Turn

15. The curve C has equation

$$x^3 \tan y = 9 \qquad 0 < y < \frac{\pi}{2}$$

(*a*) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-27x^2}{x^6 + 81}$$

(4)

(*b*) Prove that C has a point of inflection at $x = \sqrt[6]{\frac{81}{2}}$

(3)

(Total for Question 15 is 7 marks)

8. The curve C has equation

$$px^3 + qxy + 3y^2 = 26$$

where p and q are constants.

(a) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{apx^2 + bqy}{qx + cy}$$

where a, b and c are integers to be found.

(4)

Given that

- the point $P(-1, -4)$ lies on C
- the normal to C at P has equation $19x + 26y + 123 = 0$

(b) find the value of p and the value of q.

(5)

Your Turn

8. The curve C has equation

$$px^3 + qxy + 2y^2 = 28$$

where p and q are constants.

(a) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{apx^2 + bqy}{qx + cy}$$

where a, b and c are integers to be found.

(4)

Given that

- the point P $(-1, -5)$ lies on C
- the normal to C at P has equation $8x + 13y + 73 = 0$

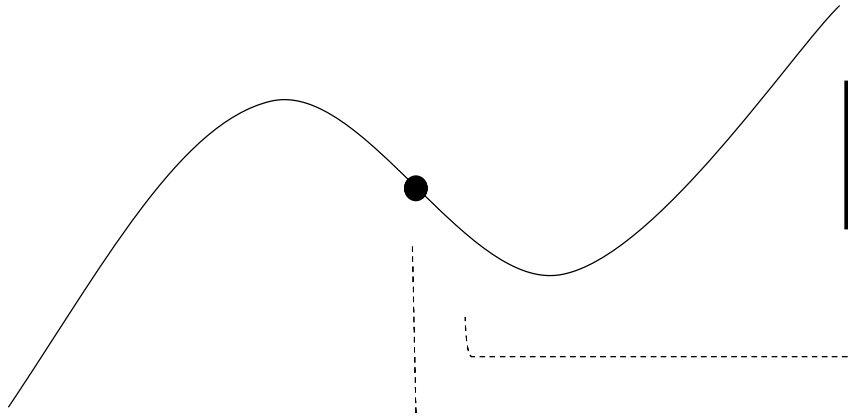
(b) find the value of p and the value of q. **(5)**

(Total for Question 8 is 9 marks)

9.9) Using second derivatives

Using the second derivative:

In Year 12 you may remember that a point of inflection was where the concavity of a curve changes, i.e. concave to convex or vice versa, or informally, ‘swerving one way to swerving the other’.



Memorisation tip: Mentally visualise a positive cubic (i.e. uphill) and put ‘concave’ and ‘convex is alphabetical order.

In this region the gradient, although initially negative, is still also increasing. The curve is **swerving left**.

$f(x)$ convex when $f''(x) \geq 0$

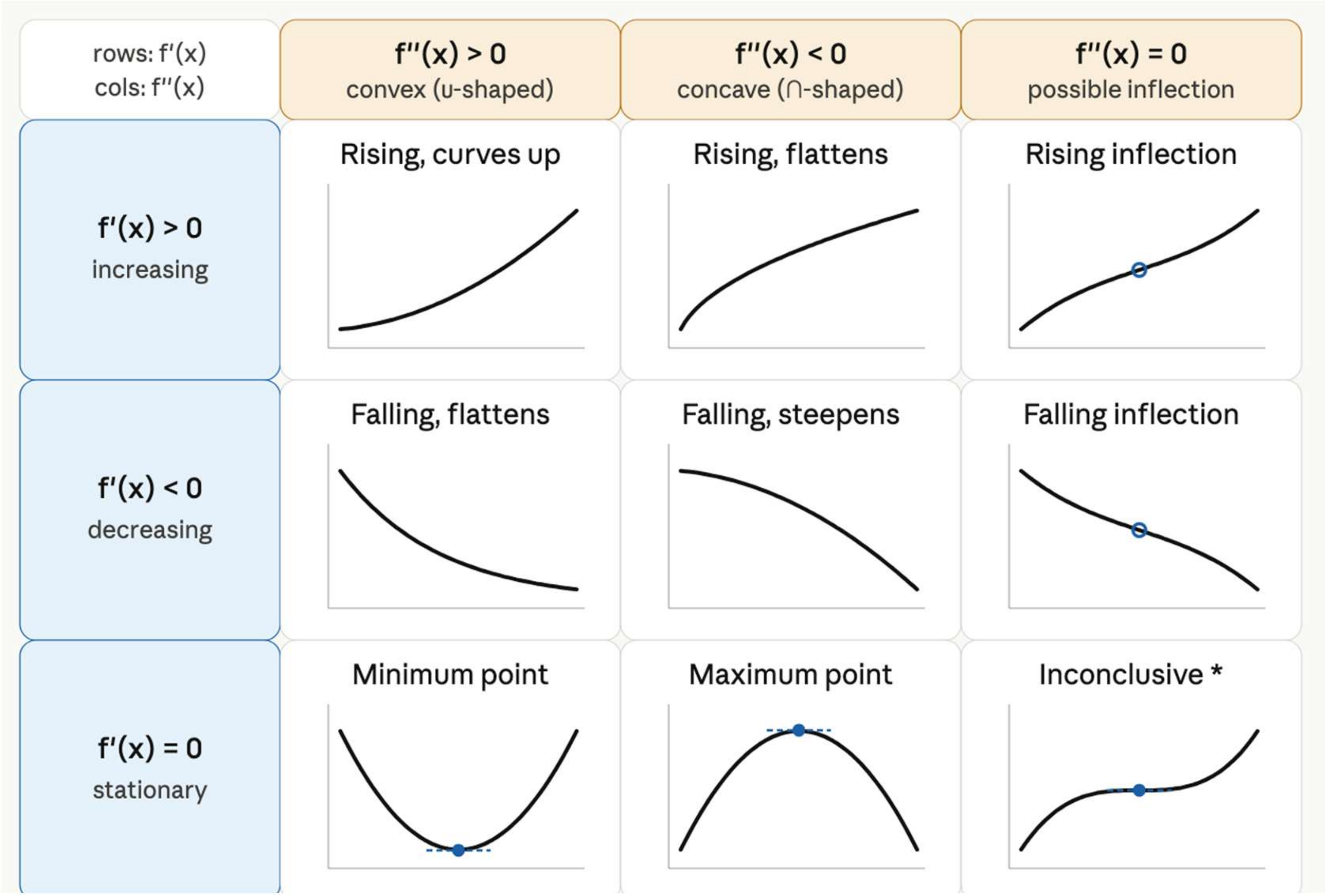
We say the curve is concave when it is **swerving right**: we can see the gradient is always decreasing, i.e. the change in the gradient is negative.

$f(x)$ concave when $f''(x) \leq 0$

At point of inflection, the gradient is not changing, i.e. the gradient of the gradient is 0. Note that points of inflection (as with this example) are not necessarily stationary points.

Point of inflection when $f''(x) = 0$

Full Summary



Worked Example

Show that the function is convex for all real values of x:

$f(x) = e^{3x} + x^2$ $\quad\quad\quad\quad$ $g(x) = e^{4x} + x^4$

T.259: Q 1-2, P.77: Q1-2

Worked Example

Determine if there is a point of inflection on the curve with equation $y = (x - 3)^4$

Worked Example

A curve C has equation

$$y = \frac{1}{4}x^2 \ln x - 3x + 7, x > 0$$

Find where C is convex

T.259: Q 3+, P.77: Q3+

5. The curve C has equation

$$y = 5x^4 - 24x^3 + 42x^2 - 32x + 11 \qquad x \in \mathbb{R}$$

(a) Find

(i) $\dfrac{dy}{dx}$

(ii) $\dfrac{d^2y}{dx^2}$

(3)

(b) (i) Verify that C has a stationary point at $x = 1$

(ii) Show that this stationary point is a point of inflection, giving reasons for your answer.

(4)

Your Turn

5. The curve C has equation

$$y = 3x^4 - 28x^3 + 96x^2 - 144x + 15 \qquad x \in \mathbb{R}$$

(*a*) Find

(i) $\dfrac{dy}{dx}$

(ii) $\dfrac{d^2y}{dx^2}$

(3)

(*b*) (i) Verify that C has a stationary point at $x = 2$

(ii) Show that this stationary point is a point of inflection, giving reasons for your answer.

(4)

(Total for Question 5 is 7 marks)

9.10) Rates of change

This is an application of the chain rule as follows:

1. Given a context where multiple (usually three) variables are related, use the chain rule to relate one rate of change to another
2. Rate often refers to /(time) i.e. $\frac{d}{dt}$, unless otherwise stated
3. Be careful with units!

Everything follows this structure:

Rate you want = (rate of change wrt intermediate) × (given rate)

Examples:

Scenario	Chain rule
Air is pumped into a spherical balloon.	$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$
Temperature depends on radius	$\frac{dT}{dt} = \frac{dT}{dr} \cdot \frac{dr}{dt}$
Volume of a shape with respect to radius	$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

Notes

Worked Example

Given that the area of a circle A cm^2 is related to its radius r cm by the formula $A = \pi r^2$, and that the rate of change of its radius in cm s^{-1} is given by $\frac{dr}{dt} = 5$, find $\frac{dA}{dt}$ when $r = 3$.

T.263: Q 1-5, P.77: Q1-4

Worked Example

Atmospheric pressure decreases as altitude increases. The rate at which atmospheric pressure decreases is proportional to the current air pressure. Write down a differential equation for the rate of change of the atmospheric pressure.

Worked Example

A metal bar is heated from a starting temperature. It is noted that the rate of gain of temperature of the metal bar is proportional to the difference between the temperature of the metal bar and its surroundings. Write a differential equation to represent this situation.

Worked Example

Determine the rate of change of the area A of a circle when the radius $r = 7$cm, given that the radius is changing at a rate of $4\ cm\ s^{-1}$.

Worked Example

A cube is expanding uniformly as it is heated. At time t seconds, the length of each edge is x cm and the volume of the cube is $V\ cm^3$.

Given that the volume increases at a constant rate of $0.024\ cm^3\ s^{-1}$, find the rate of increase of the total surface area of the cube in $cm^2\ s^{-1}$ when $x = 4$

Worked Example

A right circular cylindrical metal rod is expanding as it is heated. After t seconds the radius of the rod is x cm and the length of the rod is $5x$ cm. The cross-sectional area of the rod is increasing at the constant rate of $0.016\ cm^2\ s^{-1}$. Find the rate of increase of the volume of the rod when $x = 4$

T.263: Q 6+, P.78: Q5+

A company makes toys for children.

Figure 5 shows the design for a solid toy that looks like a piece of cheese.

The toy is modelled so that

- face ABC is a sector of a circle with radius r cm and centre A
- angle $BAC = 0.8$ radians
- faces ABC and DEF are congruent
- edges AD, CF and BE are perpendicular to faces ABC and DEF
- edges AD, CF and BE have length h cm

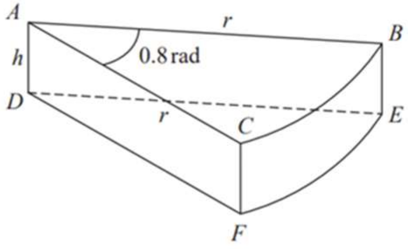


Figure 5

Given that the volume of the toy is $240\,\text{cm}^3$

(a) show that the surface area of the toy, $S\,\text{cm}^2$, is given by

$$S = 0.8r^2 + \frac{1680}{r}$$

making your method clear.

(4)

Using algebraic differentiation,

(b) find the value of r for which S has a stationary point.

(4)

(c) Prove, by further differentiation, that this value of r gives the minimum surface area of the toy.

(2)

HOME

15.

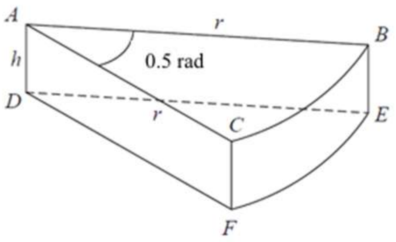


Figure 5

Harminder models a slice of the cheesecake she has made.

Figure 5 shows the outline of the cheesecake.

The slice is modelled so that

- face ABC is a sector of a circle with radius r cm and centre A
- angle $BAC = 0.5$ radians
- faces ABC and DEF are congruent
- edges AD, CF and BE are perpendicular to faces ABC and DEF
- edges AD, CF and BE have length h cm

Given that the volume of the slice of cheesecake is 125 cm^3

(a) show that the surface area of the slice, S cm^2, is given by

$$S = \frac{r^2}{2} + \frac{1250}{r}$$

making your method clear.

(4)

Using algebraic differentiation,

(b) find the value of r for which S has a stationary point.

(4)

(c) Prove, by further differentiation, that this value of r gives the minimum surface area of the slice.

(2)

(Total for Question 15 is 10 marks)

Extract from Formulae book

Trigonometric identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad (A \pm B \neq (k + \tfrac{1}{2})\pi)$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Small angle approximations

$$\sin\theta \approx \theta$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$

$$\tan\theta \approx \theta$$

where θ is measured in radians

Differentiation

First Principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$\mathbf{f(x)}$	$\mathbf{f'(x)}$
$\tan kx$	$k\sec^2 kx$
$\sec kx$	$k\sec kx \tan kx$
$\cot kx$	$-k\,\mathrm{cosec}^2\, kx$
$\mathrm{cosec}\, kx$	$-k\,\mathrm{cosec}\, kx \cot kx$
$\dfrac{f(x)}{g(x)}$	$\dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

A2 2018 Paper 1

Differentiation

3. $$y = \frac{5x^2 + 10x}{(x+1)^2} \qquad x \neq -1$$

(a) Show that $\dfrac{dy}{dx} = \dfrac{A}{(x+1)^n}$ where A and n are constants to be found.

(4)

(b) Hence deduce the range of values for x for which $\dfrac{dy}{dx} < 0$

(1)

Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

Question 3 (Total 5 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$\dfrac{dy}{dx} = \dfrac{(x+1)^2 \times (10x+10) - (5x^2+10x) \times 2(x+1)}{(x+1)^4}$	M1	This mark si given for an attempt to differentiate the expression for y
		A1	This mark is given for correctly differentiating the expression for y
	$\dfrac{dy}{dx} = \dfrac{(x+1) \times (10x+10) - (5x^2+10x) \times 2}{(x+1)^3}$	M1	This mark sis given for cancelling the expression through by $(x+1)$
	$\dfrac{dy}{dx} = \dfrac{10}{(x+1)^3}$	A1	This mark is given for a fully correct expression for $\dfrac{dy}{dx}$
(b)	If $A > 0$ and $n = 1, 3$ then $x < -1$	B1	This mark is given for deducing that $\dfrac{dy}{dx} < 0 \Rightarrow x < -1$.

Summary of Key Points

1 For small angles, measured in radians:
- $\sin x \approx x$
- $\cos x \approx 1 - \frac{1}{2}x^2$

2
- If $y = \sin kx$, then $\frac{dy}{dx} = k\cos kx$
- If $y = \cos kx$, then $\frac{dy}{dx} = -k\sin kx$

3
- If $y = e^{kx}$, then $\frac{dy}{dx} = ke^{kx}$
- If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$

4 If $y = a^{kx}$, where k is a real constant and $a > 0$, then $\frac{dy}{dx} = a^{kx}k\ln a$

5 The **chain rule** is: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
where y is a function of u and u is another function of x.

6 The chain rule enables you to differentiate a function of a function. In general,
- if $y = (f(x))^n$ then $\frac{dy}{dx} = n(f(x))^{n-1}f'(x)$
- if $y = f(g(x))$ then $\frac{dy}{dx} = f'(g(x))g'(x)$

7 $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

8 The **product rule**:
- If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$, where u and v are functions of x.
- If $f(x) = g(x)h(x)$ then $f'(x) = g(x)h'(x) + h(x)g'(x)$

9 The **quotient rule**:
- If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ where u and v are functions of x.
- If $f(x) = \frac{g(x)}{h(x)}$, then $f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$

10
- If $y = \tan kx$, then $\frac{dy}{dx} = k\sec^2 kx$
- If $y = \operatorname{cosec} kx$, then $\frac{dy}{dx} = -k\operatorname{cosec} kx\cot kx$
- If $y = \sec kx$, then $\frac{dy}{dx} = k\sec kx\tan kx$
- If $y = \cot kx$, then $\frac{dy}{dx} = -k\operatorname{cosec}^2 kx$

11
- If $y = \arcsin x$, then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$
- If $y = \arccos x$, then $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$
- If $y = \arctan x$, then $\frac{dy}{dx} = \frac{1}{1+x^2}$

12 If x and y are given as functions of a parameter, t: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

13
- $\frac{d}{dx}(f(y)) = f'(y)\frac{dy}{dx}$
- $\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$
- $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$

14
- The function $f(x)$ is **concave** on a given interval if and only if $f''(x) \leqslant 0$ for every value of x in that interval.
- The function $f(x)$ is **convex** on a given interval if and only if $f''(x) \geqslant 0$ for every value of x in that interval.

15 A **point of inflection** is a point at which $f''(x)$ changes sign.

16 You can use the chain rule to connect rates of change in situations involving more than two variables.

T.265: mixed ex ,P79: BSG Qs