KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 13
Pure Mathematics
P2 11 integration Booklet

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

Prior knowledge check

Prior knowledge check

1 Differentiate:

a $(2x-7)^6$ **b** $\sin 5x$

c $e^{\frac{x}{3}}$

← Sections 9.1, 9.2, 9.3

2 Given $f(x) = 8x^{\frac{1}{2}} - 6x^{-\frac{1}{2}}$

a find $\int f(x)\,dx$

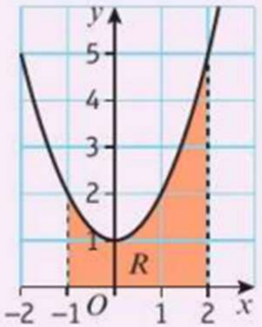
b find $\int_4^9 f(x)\,dx$

← Year 1, Chapter 13

3 Write $\dfrac{3x+22}{(4x-1)(x+3)}$ as partial fractions.

← Section 1.3

4 Find the area of the region R bounded by the curve $y = x^2 + 1$, the x-axis and the lines $x = -1$ and $x = 2$.



← Year 1, Chapter 13

11.1) Integrating standard functions

There's certain results you should be able to integrate straight off, by just thinking about the opposite of differentiation. Have a go at these by thinking about what would differentiate to the function on the left:

y	$\int y\,dx$
x^n	
e^x	
$\frac{1}{x}$	
$\cos x$	
$\sin x$	
$\sec^2 x$	
$cosec\ x \cot x$	
$cosec^2 x$	
$\sec x \tan x$	

It's vital you remember this one.

Notes

Integration is the inverse of differentiation. You can use your knowledge of derivatives to integrate familiar functions.

① $\int x^n \,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$

② $\int \mathrm{e}^x \,\mathrm{d}x = \mathrm{e}^x + c$

③ $\int \frac{1}{x} \,\mathrm{d}x = \ln|x| + c$

④ $\int \cos x \,\mathrm{d}x = \sin x + c$

⑤ $\int \sin x \,\mathrm{d}x = -\cos x + c$

⑥ $\int \sec^2 x \,\mathrm{d}x = \tan x + c$

⑦ $\int \operatorname{cosec} x \cot x \,\mathrm{d}x = -\operatorname{cosec} x + c$

⑧ $\int \operatorname{cosec}^2 x \,\mathrm{d}x = -\cot x + c$

⑨ $\int \sec x \tan x \,\mathrm{d}x = \sec x + c$

Watch out This is true for all values of n except -1.

Notation When finding $\int \frac{1}{x}\,\mathrm{d}x$ it is usual to write the answer as $\ln|x| + c$. The modulus sign removes difficulties that could arise when evaluating the integral for negative values of x.

Links For example, if $y = \cos x$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = -\sin x$. This means that $\int (-\sin x)\,\mathrm{d}x = \cos x + c$ and hence $\int \sin x \,\mathrm{d}x = -\cos x + c$. ← Section 9.1

Worked Example

Find:

$$\int 3\sin x - \frac{4}{x^2} + \sqrt[3]{x}\, dx$$

Worked Example

Find:

$$\int \frac{\sin x}{\cos^2 x}\ dx$$

T.295 11A Q1, P.90 Q1-3

Worked Example

Given that

$$\int_a^{5a} \frac{3x-1}{x}\, dx = \ln 2,$$

find the exact value of a.

T.295 11A Q2+, P.90 Q4+

11.2) Integrating $f(ax+b)$

For any expression where inner function is $ax+b$, integrate as before and $\div a$.

$$\int f'(ax+b)dx = \frac{1}{a}f(ax+b) + C$$

Worked Example

$\int (3x-1)^2$

$\int (3x-1)\,dx$

$\int \frac{1}{3x-1}\,dx$

$\int \frac{1}{(3x-1)^2}\,dx$

Worked Example

$\int \sin(6x+1)\ dx$

$\int -\sin\left(\frac{x}{5}-2\right)\ dx$

$\int \sin(3-4x)\ dx$

Worked Example

$\int \cos(6x + 1)\ dx$

$\int -\cos\left(\frac{x}{5} - 2\right)\ dx$

$\int \cos(3 - 4x)\ dx$

Worked Example

$\int \frac{1}{6x-1}\,dx$

$\int \frac{1}{\frac{1}{5}x+2}\,dx$

$\int \frac{1}{3-4x}\,dx$

Worked Example

$\int \sec^2(2x-3)\,dx$

$\int 6\sec^2(5-4x)\,dx$

Worked Example

$\int \sec 5x \tan 5x \, dx$

$\int \sec \frac{x}{4} \tan \frac{x}{4} \, dx$

Worked Example

$$\int (e^x - 1)^3 \, dx$$

Exercise 11B

1 a $\int \sin(2x+1)\,dx =$?

c $\int 4e^{x+5}\,dx =$?

e $\int cosec^2 3x\,dx =$?

f $\int \sec 4x \tan 4x\,dx =$?

g $\int 3\sin\left(\frac{1}{2}x+1\right)dx =$?

h $\int cosec\,2x \cot 2x\,dx =$?

2 a $\int e^{2x} - \frac{1}{2}\sin(2x-1)\,dx$

$=$?

b $\int (e^x+1)^2\,dx =$?

c $\int \sec^2 2x\,(1+\sin 2x)\,dx =$?

d $\int \frac{3-2\,co\left(\frac{1}{2}x\right)}{\sin^2\left(\frac{1}{2}x\right)}\,dx =$?

e $\int e^{3-x} + \sin(3-x) + \cos(3-x)\,dx$

$=$?

3 a $\int \frac{1}{2x+1}\,dx =$?

b $\int \frac{1}{(2x+1)^2}\,dx =$?

c $\int (2x+1)^2\,dx =$?

d $\int \frac{3}{4x-1}\,dx =$?

f $\int \frac{3}{(1-4x)^2}\,dx =$?

h $\int \frac{3}{(1-2x)^3}\,dx =$?

j $\int \frac{5}{3-2x}\,dx =$?

4 a $\int 3\sin(2x+1) + \frac{4}{2x+1}\,dx$

$=$?

c $\int \frac{1}{\sin^2 2x} + \frac{1}{1+2x} + \frac{1}{(1+2x)^2}\,dx$

$=$?

11B

E/P **6** Given $\int_3^b (2x-6)^2\,\mathrm{d}x = 36$, find the value of b. **(4 marks)**

E/P **7** Given $\int_{e^2}^{e^8} \frac{1}{kx}\,\mathrm{d}x = \frac{1}{4}$, find the value of k. **(4 marks)**

E/P **8** Given $\int_{\frac{\pi}{4k}}^{\frac{\pi}{3k}} (1 - \pi \sin kx)\,\mathrm{d}x = \pi(7 - 6\sqrt{2})$, find the exact value of k. **(7 marks)**

Problem-solving

Calculate the value of the indefinite integral in terms of k and solve the resulting equation.

Challenge

Given $\int_5^{11} \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln\left(\frac{41}{17}\right)$, and that a and b are integers with $0 < a < 10$, find two different pairs of values for a and b.

11.2 Integrating f($ax + b$)

1 **a** Differentiate:

i $(4x-5)^4$ **ii** $(3-2x)^6$

b Find:

i $\int 16(4x-5)^3\,\mathrm{d}x$ **ii** $\int (4x-5)^3\,\mathrm{d}x$

iii $\int \left(-12(3-2x)^5\right)\mathrm{d}x$ **iv** $\int 2(3-2x)^5\,\mathrm{d}x$

Hint Integration is the reverse process of differentiation, so use your answers to part **a** to help.

2 Integrate the following with respect to x.

a $6e^{3x}$ **b** $\cos(3x+2)$

c $(4x+1)^5$ **d** $\operatorname{cosec} 4x \cot 4x$

e $\sec^2(8x-1)$ **f** $\operatorname{cosec}^2(1-5x)$

g $\frac{5}{2x-3}$ **h** $11e^{4-x}$

Hint In general,

$\int f'(ax+b)\,\mathrm{d}x = \frac{1}{a}f(ax+b) + c$

3 Find the following integrals.

a $\int \left(\frac{1}{(4x-3)^2} - (4x-3)^2\right)\mathrm{d}x$

b $\int \sec^2 3x\left(\frac{1}{2}e^{5x}\cos^2 3x - 2\right)\mathrm{d}x$

c $\int \left(e^{2x} - \frac{1}{e^{2x}}\right)^2\mathrm{d}x$ **d** $\int \frac{5 - 3\cos 2x}{\sin^2 2x}\,\mathrm{d}x$

Hint First simplify each expression and then apply the integration rule in question 2 to each term separately.

4 Find the exact value of:

a $\int_{-2}^{0} \frac{4}{6-4x}\,\mathrm{d}x$

b $\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \operatorname{cosec}^2 2x(1 - \cos 2x)\,\mathrm{d}x$

c $\int_0^{\ln 4} (e^x - 4)^2\,\mathrm{d}x$ **d** $\int_{\frac{5\pi}{8}}^{\frac{3\pi}{4}} \frac{1}{\sin^2(2\pi - x)}\,\mathrm{d}x$

Hint Integrate each expression and then apply the limits to evaluate the integral.

E/P **5** Given that a is a positive constant and $\int_1^a (2x+1)^2\,\mathrm{d}x = 117$, find the value of a. **(4 marks)**

E/P **6** $\int_1^4 \frac{k}{2x-1}\,\mathrm{d}x = \ln 25$, where k is a real constant.

Find the value of k. **(4 marks)**

E/P **7** Find the exact value of $\int_{\ln 2}^{\ln 8} (e^x + 2)^2\,\mathrm{d}x$, giving your answer in the form $p + \ln q$, where p and q are integers. **(4 marks)**

E/P **8** Given that $\int_{\frac{\pi}{6k}}^{\frac{\pi}{3k}} \frac{\pi}{2}\cos kx\,\mathrm{d}x = \frac{9\pi}{4}(1 - \sqrt{3})$, where k is a real constant, find the value of k. **(5 marks)**

11.3) Using trigonometric identities

Given:

Trigonometric identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad (A \pm B \neq (k + \tfrac{1}{2})\pi)$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Small angle approximations

$$\sin\theta \approx \theta$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$

$$\tan\theta \approx \theta$$

where θ is measured in radians

NOT given:

Can derive from

$$sin(2A) = 2\,sin(A)\,cos(A)$$

$$\begin{aligned} cos(2A) &= cos^2(A) - sin^2(A) \\ &= 1 - 2sin^2(A) \\ &= 2cos^2(A) - 1 \end{aligned}$$

$$\tan(2A) = \frac{2\tan(A)}{1 - tan^2(A)}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \mathrm{cosec}^2\theta$$

Worked Example

Find:

$$\int (\sec x - \tan x)^2 \, dx$$

Worked Example

$$\int \sin 5x \cos 5x \; dx$$

$$\int \sin \frac{x}{4} \cos \frac{x}{4} \; dx$$

Worked Example

Find:

$$\int (\sin x - \cos x)^2 dx$$

Worked Example

Find:

$$\int (\cos 2x + 1)^2 dx$$

Worked Example

Find:

$$\int \frac{(1 + \sin x)^2}{\cos^2 x} dx$$

Worked Example

Find:

$$\int \frac{\cos 2x}{\sin^2 x}\,dx$$

T.300 11C Qs 1-2, P.91 11.3 Qs 1-3

Worked Example

Show that:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^2 x \ dx = \frac{\pi}{24} + \frac{2-\sqrt{3}}{8}$$

Worked Example

Find:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 3x \; dx$$

Worked Example

Find:

$$\int \sin^4 x \ dx$$

T.300 11C Qs 3+, P.91 11.3 Qs 4+

11.4) Reverse chain rule

There's certain more complicated expressions which look like the result of having applied the chain rule. I call this process '**consider then scale**':

1. **Consider some expression that will differentiate to something similar to it.**
2. **Differentiate, and adjust for any scale difference.**

$$\int x(x^2+5)^3\,dx \qquad \int \cos x \sin^2 x\,\,dx \qquad \int \frac{2x}{x^2+1}\,dx$$

Notes

- **To integrate expressions of the form $\int k\frac{f'(x)}{f(x)}dx$, try $\ln|f(x)|$ and differentiate to check, and then adjust any constant.**

Watch out You can't use this method to integrate a function such as $\frac{1}{x^2+3}$ because the derivative of x^2+3 is $2x$, and the top of the fraction does not contain an x term.

You can use a similar method with functions of the form $kf'(x)(f(x))^n$.

Quickfire

In your head!

Tip: If there's as power around the whole denominator, DON'T use ln: re-express the expression as a product.
e.g. $x(x^2+5)^{-3}$

$$\int \frac{4x^3}{x^4-1}\,dx =$$

$$\int \frac{\cos x}{\sin x+2}\,dx =$$

$$\int \cos x \; e^{\sin x}\,dx =$$

$$\int \cos x\,(\sin x-5)^7\,dx =$$

$$\int x^2(x^3+5)^7 =$$

Not in your head…

$$\int \frac{x}{(x^2+5)^3}\,dx =$$

Worked Example

Find:

$$\int \frac{3x^2 + 5}{\sqrt{x^3 + 5x - 2}}\, dx$$

Worked Example

Find:

$$\int \frac{\sec^2 x}{\tan x - 3}\,dx$$

Worked Example

$\int \sin x \ e^{\cos x} \, dx$

$\int \sec^2 x \ e^{\tan x} \, dx$

Worked Example

$\int \sin x\,(\cos x - 1)^5\,dx$

$\int \sec^2 x\,(\tan x + 3)^6\,dx$

Worked Example

Find:

$$\int \sec^2 x \tan x \ dx$$

Worked Example

Find:

$$\int \frac{cosec^2 x}{(3-\cot x)^4}\,dx$$

Worked Example

Find:

$$\int \frac{cosec^2 x}{(3 - \cot x)^4}\,dx$$

T.302 11D Qs all, P. 92 11.3 all

11.5) Integration by substitution

For some integrations involving a complicated expression, we can make a substitution to turn it into an equivalent integration that is simpler. We wouldn't be able to use 'reverse chain rule' on the following:

Q Use the substitution $u = 2x + 5$ to find $\int x\sqrt{2x+5}\ dx$

The aim is to completely remove any reference to x, and replace it with u. We'll have to work out x and dx so that we can replace them.

STEP 1: Using substitution, work out x and dx (or variant)

$$\frac{du}{dx} = 2 \quad \rightarrow \quad dx = \frac{1}{2}du$$

$$x = \frac{u-5}{2}$$

Tip: Be careful about ensuring you reciprocate when rearranging.

STEP 2: Substitute these into expression.

$$\int x\sqrt{2x+5}\ dx = \int \frac{u-5}{2}\sqrt{u}\frac{1}{2}du = \frac{1}{4}\int \sqrt{u}(u-5)\ du$$

$$= \frac{1}{4}\int u^{\frac{3}{2}} - 5u^{\frac{1}{2}}$$

Tip: If you have a constant factor, factor it out of the integral.

STEP 3: Integrate simplified expression.

$$= \frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{10}{3}u^{\frac{3}{2}}\right) + C$$

STEP 4: Write answer in terms of x.

$$= \frac{(2x+5)^{\frac{5}{2}}}{10} - \frac{5(2x+5)^{\frac{3}{2}}}{6} + C$$

Notes

How can we tell what substitution to use?

In Edexcel you will usually be given the substitution!
However in some other exam boards, and in STEP, you often aren't.
There's no hard and fast rule, but it's often helpful to replace to replace expressions inside roots, powers or the denominator of a fraction.

Sensible substitution:

$$\int \cos x \sqrt{1 + \sin x}\, dx \qquad \boldsymbol{u} =$$

$$\int 6x\, e^{x^2}\, dx \qquad \boldsymbol{u} =$$

$$\int \frac{xe^x}{1+x}\, dx \qquad \boldsymbol{u} =$$

$$\int e^{\frac{1-x}{1+x}}\, dx \qquad \boldsymbol{u} =$$

Worked Example

Find:

$$\int \frac{3\sin 2x}{2+\sin x}\,dx$$

using the substitution $u = 2 + \sin x$

T.306 11E Qs 1,3 P. 92 11.4 Qs1-3

Worked Example

Find:

$$\int_0^{\frac{\pi}{2}} \cos x \sin x \, (2 + \sin x)^4 \, dx$$

using the substitution $u = \sin x + 2$

Worked Example

Calculate:

$$\int_0^{\frac{\pi}{2}} \sin x \sqrt{2 + \cos x}\ dx$$

using the substitution $u = \cos x + 2$

Worked Example

Use the substitution $u = \sqrt{x} - 1$ to evaluate:

$$\int_{36}^{49} \frac{1}{\sqrt{x} - 1} \, dx$$

Worked Example

A finite region is bounded by the curve with equation $y = x^3 \ln(x^2 + 3)$, the x-axis and the lines $x = 0$ and $x = \sqrt{5}$.

Use the substitution $u = x^2 + 3$ to show that the area of R is $\frac{1}{2}\int_3^8 (u - 3) \ln u \, du$

Worked Example

Using integration by substitution, prove that:

$$-\int \frac{1}{\sqrt{1-x^2}}\,dx = \arccos x + c$$

Worked Example

Use the substitution $u = \cos x$ to evaluate

$$\int_0^{\frac{\pi}{3}} \sin^3 x \cos^2 x \, dx$$

Worked Example

Use the substitution $x = \cos u$ to evaluate

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} x^2\sqrt{1-x^2}\,dx$$

T.306 11E Qs 4+ P. 92 11.4 Qs 4+

14. (a) Use the substitution $u = 4 - \sqrt{h}$ to show that

$$\int \frac{\mathrm{d}h}{4-\sqrt{h}} = -8\ln\left|4-\sqrt{h}\right| - 2\sqrt{h} + k$$

where k is a constant

(6)

Your Turn

14. (*a*) Use the substitution $u = 6 - \sqrt{h}$ to show that

$$\int \frac{1}{6-\sqrt{h}}\,dh = -12\ln|6-\sqrt{h}| - 2\sqrt{h} + k,$$

where k is a constant

(6)

A team of scientists is studying a species of slow growing tree.

10. (a) Use the substitution $x = u^2 + 1$ to show that

$$\int_5^{10} \frac{3\,\mathrm{d}x}{(x-1)\left(3+2\sqrt{x-1}\right)} = \int_p^q \frac{6\,\mathrm{d}u}{u(3+2u)}$$

where p and q are positive constants to be found.

(4)

(b) Hence, using algebraic integration, show that

$$\int_5^{10} \frac{3\,\mathrm{d}x}{(x-1)\left(3+2\sqrt{x-1}\right)} = \ln a$$

where a is a rational constant to be found.

(6)

Your Turn

10. (*a*) Use the substitution $x = u^2 - 1$ to show that

$$\int_8^{15} \frac{5\,\mathrm{d}x}{(x+1)\left(5+2\sqrt{x+1}\right)} = \int_a^b \frac{10\,\mathrm{d}u}{u(5+2u)}$$

where a and b are positive constants to be found.

√**(4)**

(*b*) Hence, using algebraic integration, show that

$$\int_8^{15} \frac{5\,\mathrm{d}x}{(x+1)\left(5+2\sqrt{x+1}\right)} = \ln p$$

where p is a rational constant to be found.

(6)

(Total for Question 10 is 10 marks)

11.6) Integration by parts

$$\int x \cos x \ dx = ?$$

Just as the Product Rule was used to **differentiate the product** of two expressions, we can often use 'Integration by Parts' to **integrate a product.**

To integrate by parts:

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} \ dx$$

or $\int uv' \, dx = uv - \int vu' \, dx$

(or even $\int vu' \, dx = uv - \int uv' \, dx$

Notes

Abbreviation	Function Type	Examples
L	logarithmic	$\ln x, \log_{10}(x)$
I	inverse trigonometric	$\arcsin(x), \arctan x$
A	algebraic	$x^2, 5x^2+x+1$
T	trigonometric	$\sin(x), \cos(x)$
E	exponential	$\mathbf{e}^x, 8^x$

you can ignore the I and just learn it as L.A.T.E

Examples:

Integral	**u**	**V’**
$\int x\ln(x)dx$	$\ln(x)$	xdx
$\int \mathbf{e}^{2x}\cos(x)$	$\cos(x)$	$\mathbf{e}^{2x}dx$
$\int x^2\sin(x)dx$	x^2	$\sin(x)dx$

Stopping the Process:
For many integrals, integration by parts must be repeated several times before a solution is found. In this case, knowing when to stop the process is important. There are three conditions when the process should be stopped:

1.When the integral ∫ vu’ can be integrated without applying integration by parts.

2.When the integral ∫ vu’ is equal to or a multiple of the integral being evaluated. When this happens, the solution is found by solving the resulting equation algebraically.

3.When the derivative and/or integral becomes more complex with each iteration. In this case, either a bad choice for u or v’ was made, or integration by parts may not be the appropriate method for evaluating the integral.

DI METHOD

This is essentially a way to combine the rule with all the considerations discussed so far. The advantage is by tabulating the process you don't end up with long lines of algebra with multiple nested brackets and negatives.

$\int x^2 \ln x \, dx$ ① L A T E

②

	D	I
+	$\ln x$	x^2
−	$\frac{1}{x}$	$\frac{1}{3}x^3$

③ this line can be integrated when combined

④ $\longrightarrow (\ln x)\left(\frac{1}{3}x^3\right) - \int \left(\frac{1}{x}\right)\left(\frac{1}{3}x^3\right) dx$

$= \frac{1}{3}\ln x^3 - \int \frac{1}{3}x^2 \, dx$

Case A : 2nd line integrates

$\int x^2 e^x \, dx$ ① L A T E

②

	D	I
+	x^2	e^x
−	$2x$	e^x
+	2	e^x
−	0	e^x

④ $\longrightarrow = x^2e^x - 2xe^x + 2e^x$

$\therefore \int x^2 e^x \, dx = x^2e^x - 2xe^x + 2e^x + c$

③ IF you can see that the "D" will reduce to zero keep using diagonals as shown in step 4

Case B: D reduces to zero

$(I =) \int e^x \sin x \, dx$ ① L A T E

②

	D	I
+	$\sin x$	e^x
−	$\cos x$	e^x
+	$-\sin x$	e^x
−		

④ $\longrightarrow I = e^x \sin x - e^x \cos x + \int (-\sin x)(e^x) dx$

$\Rightarrow I = e^x \sin x - e^x \cos x - I$

$\therefore 2I = e^x \sin x - e^x \cos x$

$\Rightarrow \int e^x \sin x \, dx = \dfrac{e^x \sin x - e^x \cos x}{2} + c$

③ IF you see the original function repeated on a row – STOP.

Case C: function repeated

Worked Example

Find:

$$\int x^2 \cos x \; dx$$

Worked Example

Find:

$$\int x^3 \ln x \ dx$$

Worked Example

Find:

$$\int e^{x} \cos x \; dx$$

T.309 Ex 11F Qs 1-3, P.94 11.6 Qs 1-3

Worked Example

Evaluate:

$$\int_1^4 \ln x \, dx$$

Worked Example

Evaluate:

$$\int_0^{\frac{\pi}{2}} x \cos x \ dx$$

Worked Example

#	Integral	Case (A, B or C)	Why? (short justification)
1	$\int x^3 e^x \, dx$		
2	$\int e^x \sin x \, dx$		
3	$\int \ln x \, dx$		
4	$\int x^2 \cos x \, dx$		
5	$\int e^x \cos x \, dx$		
6	$\int x^2 \ln x \, dx$		
7	$\int x \sin x \, dx$		
8	$\int \sin x \cos x \, dx$		

A – row integrates *B – D reduces to zero* *C – row with initial function*

T.309 Ex 11F Qs 4+, P.94 11.6 Qs 4+

12. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

Show that

$$\int_1^{e^2} x^3 \ln x \, dx = ae^8 + b$$

where a and b are rational constants to be found.

(5)

Your Turn

12. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

Show that

$$\int_{e}^{e^2} x^2 \ln x \, dx = \frac{1}{9}e^3\left(Ae^3 + B\right)$$

where A and B are rational constants to be found.

(5)

(Total for Question 12 is 5 marks)

11.7) Partial fractions

Recall:

- **To integrate expressions of the form $\int k\frac{f'(x)}{f(x)}\,dx$, try $\ln|f(x)|$ and differentiate to check, and then adjust any constant.**

Watch out You can't use this method to integrate a function such as $\frac{1}{x^2+3}$ because the derivative of x^2+3 is $2x$, and the top of the fraction does not contain an x term.

You can use a similar method with functions of the form $kf'(x)(f(x))^n$.

This can be applied to an integral which can be written as partial fractions, as generally:

$$\int \frac{a}{ax+b}\,dx = \ln(ax+b)+c$$

Worked Example

Find:

$$\int \frac{x+5}{(x-1)(x+2)}\,dx$$

Worked Example

Find:

$$\int \frac{3x + 15 - 4x^2}{(2x+1)(x-2)^2}\, dx$$

T.312 Ex 11G Qs 1-3, P.95 11.7 Qs 1-4

Worked Example

Evaluate:

$$\int_0^2 \frac{8x^2 + 34x + 20}{(2x+1)(x+1)(x+3)}$$

Worked Example

Find:

$$\int \frac{4x^2 - 2x - 18}{4x^2 - 9}\, dx$$

T.312 Ex 11G Qs 4+, P.95 11.7 Qs 5+

6. (a) Given that

$$\frac{x^2 + 8x - 3}{x + 2} \equiv Ax + B + \frac{C}{x + 2} \qquad x \in \mathbb{R} \;\; x \neq -2$$

find the values of the constants A, B and C

(3)

(b) Hence, using algebraic integration, find the exact value of

$$\int_0^6 \frac{x^2 + 8x - 3}{x + 2}\,dx$$

giving your answer in the form $a + b\ln 2$ where a and b are integers to be found.

(4)

Your Turn

6. (*a*) Given that

$$\frac{x^2+10x-5}{x+4} \equiv Ax+B+\frac{C}{x+4} \qquad x \in \mathbb{R}\ x \neq -4$$

find the values of the constants A, B and C

(3)

(*b*) Hence, using algebraic integration, find the exact value of

$$\int_0^8 \frac{x^2+10x-5}{x+4}\,\mathrm{d}x$$

giving your answer in the form $a + b\ln 3$ where a and b are integers to be found.

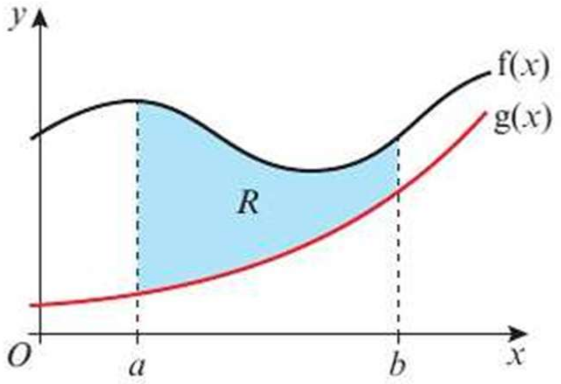
(4)

(Total for Question 6 is 7 marks)

11.8) Finding areas

- **The area bounded by two curves can be found using integration:**

$$\text{Area of } R = \int_a^b (\mathrm{f}(x) - \mathrm{g}(x))\,\mathrm{d}x = \int_a^b \mathrm{f}(x)\,\mathrm{d}x - \int_a^b \mathrm{g}(x)\,\mathrm{d}x$$



Worked Example

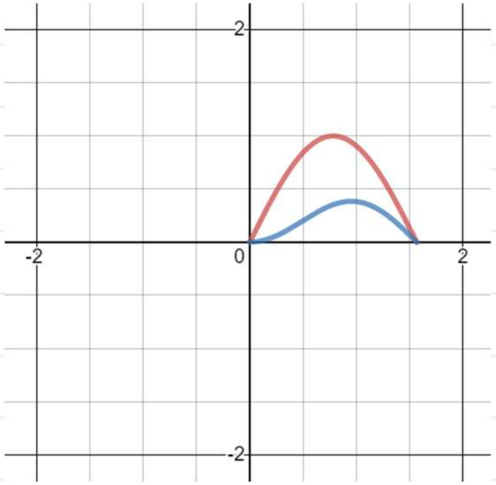
A finite region is bound by the curve

$y = \frac{3}{\sqrt{9+4x}}$, the x-axis, and the lines $x = 0$ and $x = 4$. Use integration to find the area of the region.

T. 314 11H Qs1-6, P.96 11.8 Qs 1-2

Worked Example

A finite region is bound between the curves $y = \sin 2x$ and $y = cos\, x \sin^2 x$ where $0 \leq x \leq \frac{\pi}{2}$. Use integration to find the area of the region.



T. 314 11H Qs7+, P.96 11.8 Qs 3+

11.8+) Finding areas: Areas under parametric curves

Area: $\int y\,dx = \int y\frac{dx}{dt}\,dt$

Notes

Worked Example

Determine the area bound between the curve with parametric equations $x = t^2$ and $y = t - 1$, the x-axis, and the lines $x = 0$ and $x = 5$.

Worked Example

The curve C has parametric equations

$$x = t(2+t), \qquad y = \frac{1}{2+t}, \qquad t \geq 0$$

Find the exact area of the region, bounded by C, the x-axis and the lines $x = 0$ and $x = 8$.

Worked Example

The curve C has parametric equations

$$x = 1 - \frac{1}{4}t, \qquad y = 4^t - 1, \qquad t \geq 0$$

A finite region is bounded by the curve C, the x-axis and the line $x = -1$. Find the exact area of this region.

Exercise 11.X

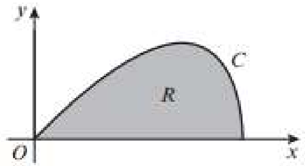
This exercise is not in the current version of the Pearson textbooks as the content was added later. I have temporarily included the exercise subsequently produced by Pearson.

(P) **1** The curve C has parametric equations $x = t^3$, $y = t^2$, $t \geqslant 0$. Show that the exact area of the region bounded by the curve, the x-axis and the lines $x = 0$ and $x = 4$ is $k\sqrt[3]{2}$, where k is a rational constant to be found.

(E/P) **2** The curve C has parametric equations

$$x = \sin t,\ y = \sin 2t,\ 0 \geqslant t \geqslant \frac{\pi}{2}$$

The finite region R is bounded by the curve and the x-axis.
Find the exact area of R. **(6 marks)**

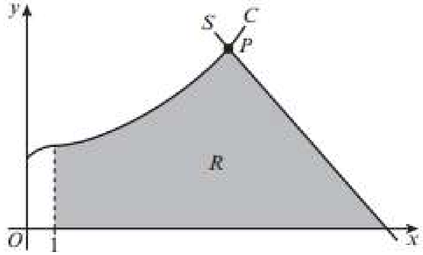


(E/P) **3** This graph shows part of the curve C with parametric equations $x = (t + 1)^2$, $y = \frac{1}{2}t^3 + 3$, $t \geqslant -1$
P is the point on the curve where $t = 2$.
The line S is the normal to C at P.

a Find an equation of S. **(5 marks)**

The shaded region R is bounded by C, S, the x-axis and the line with equation $x = 1$.

b Using integration, find the area of R. **(5 marks)**



ANSWERS

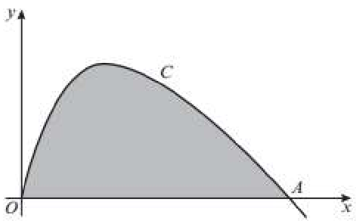
1 ?
2 ?
3 ?
4 ?
5 ?
6 ?
7 ?

(E/P) **4** The diagram shows the curve C with parametric equations

$$x = 3t^2,\ y = \sin 2t,\ t \geqslant 0.$$

a Write down the value of t at the point A where the curve crosses the x-axis. **(1 mark)**

b Find, in terms of π, the exact area of the shaded region bounded by C and the x-axis. **(6 marks)**



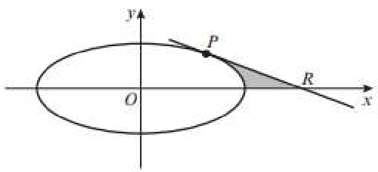
(E/P) **5** The curve shown has parametric equations

$$x = 5\cos\theta,\ y = 4\sin\theta,\ 0 \leqslant \theta < 2\pi$$

a Find the gradient of the curve at the point P at which $\theta = \frac{\pi}{4}$ **(3 marks)**

b Find an equation of the tangent to the curve at the point P. **(3 marks)**

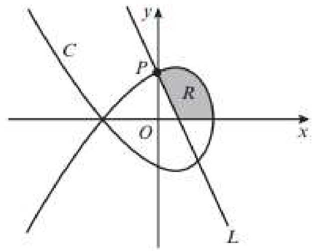
c Find the exact area of the shaded region bounded by the tangent PR, the curve and the x-axis. **(6 marks)**



(E/P) **6** The curve C has parametric equations

$$x = 1 - t^2,\ y = 2t - t^3,\ t \in \mathbb{R}$$

The line L is a normal to the curve at the point P where the curve intersects the positive y-axis. Find the exact area of the region R bounded by the curve C, the line L and the x-axis, as shown on the diagram. **(7 marks)**



(E/P) **7** The curve shown in the diagram has parametric equations

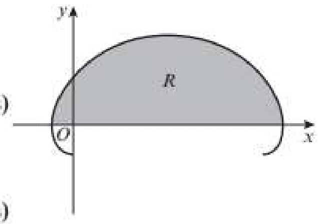
$$x = t - 2\sin t,\ y = 1 - 2\cos t,\ 0 \leqslant t \leqslant 2\pi$$

a Show that the curve crosses the x-axis where $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$ **(3 marks)**

The finite region R is enclosed by the curve and the x-axis, as shown shaded in the diagram.

b Show that the area R is given by $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2 \, dt$ **(3 marks)**

c Use this integral to find the exact value of the shaded area. **(4 marks)**



The curve shown in Figure 3 has parametric equations

$$x = 6\sin t \qquad y = 5\sin 2t \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

The region R, shown shaded in Figure 3, is bounded by the curve and the x-axis.

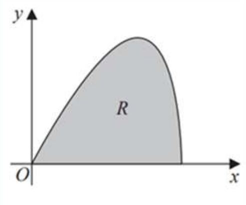


Figure 3

(a) (i) Show that the area of R is given by $\displaystyle\int_0^{\frac{\pi}{2}} 60\sin t\cos^2 t \, dt$

(3)

(ii) Hence show, by algebraic integration, that the area of R is exactly 20

(3)

Part of the curve is used to model the profile of a small dam, shown shaded in Figure 4.
Using the model and given that

- x and y are in metres
- the vertical wall of the dam is 4.2 metres high
- there is a horizontal walkway of width MN along the top of the dam

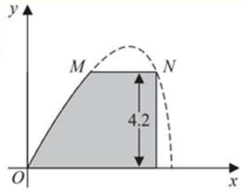


Figure 4

(b) calculate the width of the walkway.

(5)

12.

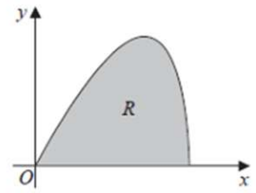


Figure 3

The curve shown in Figure 3 has parametric equations

$$x = 5\sin t \qquad y = 9\sin 2t \qquad 0 \leq t \leq \frac{\pi}{2}$$

The region R, shown shaded in Figure 3, is bounded by the curve and the x-axis.

(*a*) (i) Show that the area of R is given by $\int_0^{\frac{\pi}{2}} 90\sin t\cos^2 t\,dt$

(3)

(ii) Hence show, by algebraic integration, that the area of R is exactly 30

(3)

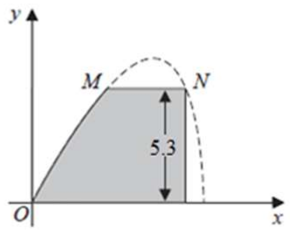


Figure 4

Part of the curve is used to model the profile of a small dam, shown shaded in Figure 4.
Using the model and given that

- x and y are in metres
- the vertical wall of the dam is 5.3 metres high
- there is a horizontal walkway of width MN along the top of the dam

(*b*) calculate the width of the walkway.

(5)

(Total for Question 12 is 11 marks)

16.

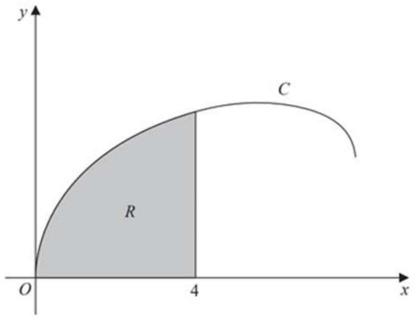


Figure 6

Figure 6 shows a sketch of the curve C with parametric equations

$$x = 8\sin^2 t \qquad y = 2\sin 2t + 3\sin t \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

The region R, shown shaded in Figure 6, is bounded by C, the x-axis and the line with equation $x = 4$

(a) Show that the area of R is given by

$$\int_0^a \left(8 - 8\cos 4t + 48\sin^2 t\cos t\right)\mathrm{d}t$$

where a is a constant to be found.

(5)

(b) Hence, using algebraic integration, find the exact area of R.

(4)

Your Turn

16.

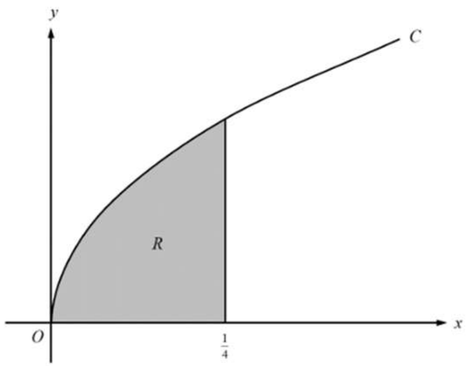


Figure 6

Figure 6 shows a sketch of the curve C with parametric equations

$$x = \cos^2 t \qquad y = 3\cos 2t + 2\cos t + 3 \qquad \frac{\pi}{4} \leqslant t \leqslant \frac{\pi}{2}$$

The region R, shown shaded in Figure 6, is bounded by C, the x-axis and the line with equation $x = \frac{1}{4}$

(a) Show that the area of R is given by

$$\int_{\frac{\pi}{2}}^{a} \left(-\frac{3}{2}\sin 4t - 4\cos^2 t \sin t - 3\sin 2t \right) \mathrm{d}t$$

where a is a constant to be found.

(5)

(b) Hence, using algebraic integration, find the exact area of R.

(4)

(Total for Question 16 is 9 marks)

11.9) The trapezium rule

In general:

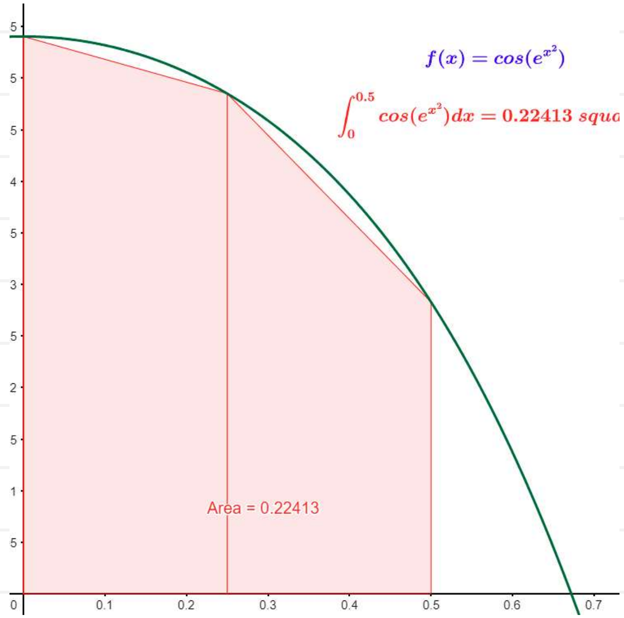
width of each trapezium

$$\int_a^b y\,dx \approx \frac{h}{2}\left(y_0 + 2(y_1 + \cdots + y_{n-1}) + y_n\right)$$

Area under curve

is approximately

[Trapezium rule for concave downwards, – GeoGebra]()



Worked Example

$$I = \int_0^{\frac{\pi}{3}} \sec x \ dx$$

Use the trapezium rule with two strips to estimate I.

Worked Example

$$I = \int_0^{\frac{\pi}{2}} \sin x \; dx$$

a) Use the trapezium rule with four strips to estimate I
b) State, with a reason, whether your approximation is an underestimate or an overestimate
c) Find the percentage error of your estimate to the exact value of I
d) Give one way the trapezium rule can be used to give a more accurate approximation

T.319 Ex 11I , P.98 11.9

1 The table below shows corresponding values of x and y for $y = \sqrt{\frac{x}{1+x}}$

The values of y are given to 4 significant figures.

x	0.5	1	1.5	2	2.5
y	0.5774	0.7071	0.7746	0.8165	0.8452

(a) Use the trapezium rule, with all the values of y in the table, to find an estimate for

$$\int_{0.5}^{2.5} \sqrt{\frac{x}{1+x}}\,dx$$

giving your answer to 3 significant figures.

(3)

(b) Using your answer to part (a), deduce an estimate for $\int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}}\,dx$

(1)

Given that

$$\int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}}\,dx = 4.535 \text{ to 4 significant figures}$$

(c) comment on the accuracy of your answer to part (b).

(1)

Your Turn

1 The table below shows corresponding values of x and y for $y = \sqrt{\frac{2x}{5+x}}$

The values of y are given to 4 significant figures.

x	0.75	1.5	2.25	3	3.75
y	0.5108	0.6794	0.7878	0.8660	0.9258

(*a*) Use the trapezium rule, with all the values of y in the table, to find an estimate for

$$\int_{0.75}^{3.75} \sqrt{\frac{2x}{5+x}}\,dx$$

giving your answer to 3 significant figures.

(3)

(*b*) Using your answer to part (a), deduce an estimate for $\int_{0.75}^{3.75} \sqrt{\frac{32x}{5+x}}\,dx$

(1)

Given that

$$\int_{0.75}^{3.75} \sqrt{\frac{32x}{5+x}}\,dx = 9.196 \text{ to 4 significant figures}$$

(*c*) comment on the accuracy of your answer to part (b).

(1)

(Total for Question 1 is 5 marks)

11.10) Solving differential equations

Differential equations are equations involving a mix of variables and derivatives, e.g. $\boldsymbol{y}$, $\boldsymbol{x}$ and $\boldsymbol{\frac{dy}{dx}}$.

'Solving' these equations means to get y in terms of x (with no $\frac{dy}{dx}$).

From year 1: if $\frac{dy}{dx} = f(x)$, then: $y = \int f(x)dx$

Now: if $\frac{dy}{dx} = f(x)g(y)$, then: $\int \frac{1}{g(y)} dy = \int f(x)dx$

This method of solving differential equations is known as **separating variables**

11.10) Solving differential equations

Split the following up into the form $\int \frac{1}{g(y)} dy = \int f(x) dx$ <u>IF POSSIBLE!</u>

1. $\frac{dy}{dx} = xy$

2. $\frac{dy}{dx} = x + y$

3. $\frac{dy}{dx} = (x^2 + 1)y^3$

4. $\frac{dy}{dx} = \sin(xy)$

5. $\frac{dy}{dx} = \sin(x)\sin(y)$

6. $\frac{dy}{dx} = x + y^2$

7. $\frac{dy}{dx} = e^x \cos y$

8. $\frac{dy}{dx} = xy + y$

Key Points

- Get y on to LHS by dividing (possibly factorising first).
- If after integrating you have ln on the RHS, make your constant of integration $\ln k$.
- Be sure to combine all your ln's together, e.g.:

$$2\ln|x+1| - \ln|x| \quad \rightarrow \quad \ln\left|\frac{(x+1)^2}{x}\right|$$

- Sub in boundary conditions to work out your constant – better to do sooner rather than later.
- Exam questions - partial fractions combined with differential equations.

Worked Example

Find the general solution to:

$$(1 - x^2)\frac{dy}{dx} = x \cot y$$

T. 324 Ex 11J Q1 , P. 100 11.10 Q 1,2

Worked Example

Find the particular solution to:

$$\frac{dy}{dx} = -\frac{3(y+2)}{(2x-1)(x-2)}$$

given that $x = 4$ when $y = 5$

Worked Example

Find the particular solution to:

$$\frac{dy}{dx} = -\frac{5}{y \sin^2 x}$$

given that $y = 4$ at $x = \frac{\pi}{4}$

Worked Example

Find the particular solution to:

$$\frac{dy}{dx} = xy\cos x$$

given that $y = 1$ at $x = \frac{\pi}{2}$

T. 324 Ex 11J Q2+, P. 100 11.10 Q 3+

11.11) Modelling with differential equations

Worked Example

The rate of increase of a population P of microorganisms at time t, in hours, is given by

$$\frac{dP}{dt} = 6P, k > 0$$

Initially the population was of size 4.

a) Find a model for P in the form $P = Ae^{6t}$

b) Find, to the nearest hundred, the size of the population at time $t = 4$

c) Find the time at which the population will be 10000 times its starting value.

d) State one limitation of this model for large values of t

Worked Example

Water in a manufacturing plant is held in a large cylindrical tank of diameter 10m. Water flows out of the bottom of the tank through a tap at a rate proportional to the cube root of the volume.

(a) Show that t minutes after the tap is opened,

$\frac{dh}{dt} = -k\sqrt[3]{h}$ for some constant k.

(b) Show that the general solution of this differential equation may be written $h = (P - Qt)^{\frac{3}{2}}$, where P and Q are constants.

Initially the height of the water is 64m. 21 minutes later, the height is 27m.

(c) Find the values of the constants P and Q.

(d) Find the time in minutes when the water is at a depth of 8m.

Worked Example

A fluid reservoir initially containers 10000 litres of unpolluted fluid.
The reservoir is leaking at a constant rate of 200 litres per hour and it is suspected that contaminated fluid flows into the reservoir at a constant rate of 300 litres per day.
The contaminated fluid contains 4 grams of contaminant in every litre of fluid.
It is assumed that the contaminant instantly disperses throughout the reservoir upon entry.
Given that there are x grams of contaminant in the reservoir after t days,

(a) Show that the situation can be modelled by the differential equation

$$\frac{dx}{dt} = 1200 - \frac{2x}{100 + t}$$

(b) Hence find the number of grams of contaminant in the tank after 7 days.

(c) Explain how the model could be refined.

T. 328 Ex 11K , P. 101 11.11

14. A large spherical balloon is deflating.

At time t seconds the balloon has radius r cm and volume V cm^3

The volume of the balloon is modelled as decreasing at a constant rate.

(a) Using this model, show that

$$\frac{dr}{dt} = -\frac{k}{r^2}$$

where k is a positive constant.

(3)

Given that

- the initial radius of the balloon is 40 cm
- after 5 seconds the radius of the balloon is 20 cm
- the volume of the balloon continues to decrease at a constant rate until the balloon is empty

(b) solve the differential equation to find a complete equation linking r and t.

(5)

(c) Find the limitation on the values of t for which the equation in part (b) is valid.

(2)

Your Turn

14. A large spherical balloon is rapidly deflating. At time t seconds the balloon has radius r cm and volume V cm^3.

The volume of the balloon is modelled as decreasing at a constant rate.

(a) Using this model, show that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{k}{r^2}$$

where k is a positive constant.

(3)

Given that

- the initial radius of the balloon is 80 cm
- after 5 seconds the radius of the balloon is 30 cm
- the volume of the balloon continues to decrease at a constant rate until the balloon is empty

(b) solve the differential equation to find a complete equation linking r and t.

(5)

(c) Find the limitation on the values of t for which the equation in part (b) is valid.

(2)

(Total for Question 14 is 10 marks)

Water flows at a constant rate into a large tank.

The tank is a cuboid, with all sides of negligible thickness.

The base of the tank measures 8 m by 3 m and the height of the tank is 5 m.

There is a tap at a point T at the bottom of the tank, as shown in Figure 5.

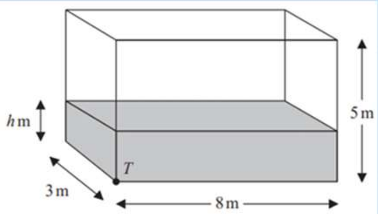


Figure 5

At time t minutes after the tap has been opened

- the depth of water in the tank is h metres
- water is flowing into the tank at a constant rate of $0.48\,m^3$ per minute
- water is modelled as leaving the tank through the tap at a rate of $0.1h\,m^3$ per minute

(a) Show that, according to the model,

$$1200\frac{dh}{dt} = 24 - 5h$$

(4)

Given that when the tap was opened, the depth of water in the tank was 2 m,

(b) show that, according to the model,

$$h = A + Be^{-kt}$$

where A, B and k are constants to be found.

(6)

Given that the tap remains open,

(c) determine, according to the model, whether the tank will ever become full, giving a reason for your answer.

(2)

Your Turn

14.

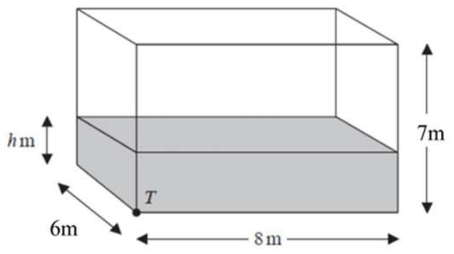


Figure 5

Water flows at a constant rate into a large tank.

The tank is a cuboid, with all sides of negligible thickness.

The base of the tank measures 8 m by 6 m and the height of the tank is 7 m.

There is a tap at a point T at the bottom of the tank, as shown in Figure 5.

At time t minutes after the tap has been opened

- the depth of water in the tank is h metres
- water is flowing into the tank at a constant rate of 0.9 m^3 per minute
- water is modelled as leaving the tank through the tap at a rate of $0.2h$ m^3 per minute

(*a*) Show that, according to the model,

$$480\frac{dh}{dt} = 9 - 2h$$

(4)

Given that when the tap was opened, the depth of water in the tank was 3 m,

(*b*) show that, according to the model,

$$h = A + B\,e^{-kt}$$

where A, B and k are constants to be found.

(6)

Given that the tap remains open,

(*c*) determine, according to the model, whether the tank will ever become full, giving a reason for your answer.

(2)

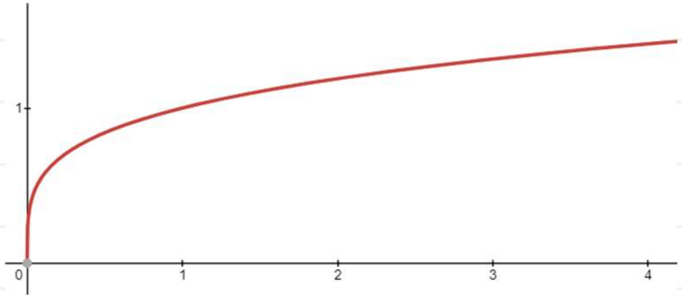
(Total for Question 14 is 12 marks)

11.12) Integration as the limit of a sum

$$\lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\delta x = \int_a^b f(x)\,dx$$

Worked Example

The diagram shows a sketch of the curve with equation $y = \sqrt[4]{x}, x > 0$.



The area under the curve may be thought of as a series of thin strips of height y and width δx.
Calculate to 4 significant figures:

$$\lim_{\delta x \to 0} \sum_{2}^{3} \sqrt[4]{x}\, \delta x$$

Worked Example

Calculate to four significant figures:

$$\lim_{\delta x \to 0} \sum_{5}^{6} \cos x \, \delta x$$

Exercise

42) Find the exact value of:

a)

$$\lim_{\delta x \to 0} \sum_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{5}{4} \sec^2 \frac{5}{6} x \, \delta x$$

b)

$$\lim_{\delta x \to 0} \sum_{x=\frac{\pi}{9}}^{\frac{2\pi}{9}} \frac{3}{8} \sec \frac{3}{2} x \tan \frac{3}{2} x \, \delta x$$

6) Find the exact value of:

a)

$$\lim_{\delta x \to 0} \sum_{x=-5}^{26} (6+x)^{\frac{3}{5}} \, \delta x$$

b)

$$\lim_{\delta x \to 0} \sum_{x=19}^{84} \frac{4}{\sqrt[4]{x-3}} \, \delta x$$

12) Find the exact value of:

a)

$$\lim_{\delta x \to 0} \sum_{x=-\frac{1}{3}}^{10} \frac{2}{5} (2+3x)^{\frac{3}{5}} \, \delta x$$

b)

$$\lim_{\delta x \to 0} \sum_{x=\frac{1}{9}}^{\frac{1}{3}} \frac{243}{(4-9x)^4} \, \delta x$$

18) Find the exact value of:

a)

$$\lim_{\delta x \to 0} \sum_{x=0}^{1} 98x(1+14x^2)^{\frac{5}{2}} \, \delta x$$

b)

$$\lim_{\delta x \to 0} \sum_{x=0}^{9} \frac{\sqrt{\sqrt{x}+1}}{\sqrt{x}} \, \delta x$$

7) Find the exact value of

$$\lim_{\delta x \to 0} \sum_{x=1}^{4} \frac{e^{2\sqrt{x}}}{\sqrt{x}} \, \delta x$$

14) Find the exact value of

$$\lim_{\delta x \to 0} \sum_{x=\frac{16}{\pi}}^{\frac{8}{\pi}} \frac{\sin\left(\frac{4}{x}\right)}{8x^2} \, \delta x$$

2022 P1 Q4

4. (a) Express $\lim_{\delta x \to 0} \sum_{x=2.1}^{6.3} \frac{2}{x} \delta x$ as an integral.

(1)

(b) Hence show that

$$\lim_{\delta x \to 0} \sum_{x=2.1}^{6.3} \frac{2}{x} \delta x = \ln k$$

where k is a constant to be found.

(2)

4. (*a*) Express $\lim_{\delta x \to 0} \sum_{x=2.4}^{9.6} \frac{5}{x} \delta x$ as an integral.

(1)

(*b*) Hence show that

$$\lim_{\delta x \to 0} \sum_{x=2.4}^{9.6} \frac{5}{x} \delta x = \ln p$$

where p is a constant to be found.

(2)

(Total for Question 4 is 3 marks)

Extract from Formulae book

Integration (+ constant)

$\mathbf{f}(x)$	$\int \mathbf{f}(x)\,\mathrm{d}x$
$\sec^2 kx$	$\frac{1}{k}\tan kx$
$\tan kx$	$\frac{1}{k}\ln\lvert\sec kx\rvert$
$\cot kx$	$\frac{1}{k}\ln\lvert\sin kx\rvert$
$\operatorname{cosec} kx$	$-\frac{1}{k}\ln\lvert\operatorname{cosec} kx + \cot kx\rvert,\quad \frac{1}{k}\ln\lvert\tan(\tfrac{1}{2}kx)\rvert$
$\sec kx$	$\frac{1}{k}\ln\lvert\sec kx + \tan kx\rvert,\quad \frac{1}{k}\ln\lvert\tan(\tfrac{1}{2}kx + \tfrac{1}{4}\pi)\rvert$

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Numerical Methods

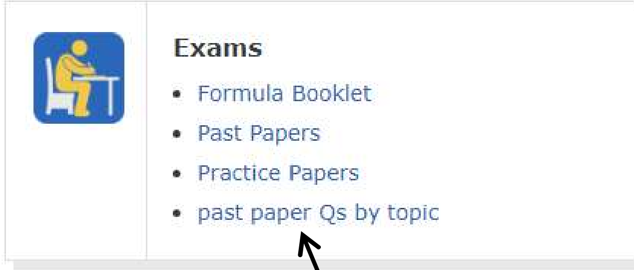
The trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \tfrac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

4. Given that a is a positive constant and

$$\int_a^{2a} \frac{t+1}{t}\,\mathrm{d}t = \ln 7$$

show that $a = \ln k$, where k is a constant to be found.

(4)



Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

Summary of Key Points

$f(x)$	How to deal with it	$\int f(x)\,dx$ (+constant)	Formula booklet?
$\sin x$	Standard result	$-\cos x$	No
$\cos x$	Standard result	$\sin x$	No
$\tan x$	In formula booklet, but use $\int \frac{\sin x}{\cos x}\,dx$ which is of the form $\int \frac{kf'(x)}{f(x)}\,dx$	$\ln\lvert\sec x\rvert$	Yes
$\sin^2 x$	For both $\sin^2 x$ and $\cos^2 x$ use identities for $\cos 2x$ $$\cos 2x = 1 - 2\sin^2 x$$ $$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$	$\frac{1}{2}x - \frac{1}{4}\sin 2x$	No
$\cos^2 x$	$$\cos 2x = 2\cos^2 x - 1$$ $$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$	$\frac{1}{2}x + \frac{1}{4}\sin 2x$	No
$\tan^2 x$	$$1 + \tan^2 x \equiv \sec^2 x$$ $$\tan^2 x \equiv \sec^2 x - 1$$	$\tan x - x$	No
$\operatorname{cosec} x$	Would use substitution $u = \operatorname{cosec} x + \cot x$, but too hard for exam.	$-\ln\lvert\operatorname{cosec} x + \cot x\rvert$	Yes
$\sec x$	Would use substitution $u = \sec x + \tan x$, but too hard for exam.	$\ln\lvert\sec x + \tan x\rvert$	Yes
$\cot x$	$\int \frac{\cos x}{\sin x}\,dx$ which is of the form $\int \frac{f'(x)}{f(x)}\,dx$	$\ln\lvert\sin x\rvert$	Yes

Summary of Key Points

$f(x)$	How to deal with it	$\int f(x)dx$ (+constant)	Formula booklet?
$\boldsymbol{cosec^2 x}$	By observation.	$-\cot x$	No!
$\boldsymbol{sec^2 x}$	By observation.	$\tan x$	Yes (but memorise)
$\boldsymbol{cot^2 x}$	$1 + \cot^2 x \equiv cosec^2 x$	$-\cot x - x$	No
$\boldsymbol{\sin 2x \cos 2x}$	For any product of sin and cos with same coefficient of x, use double angle. $\sin 2x \cos 2x \equiv \frac{1}{2}\sin 4x$	$-\frac{1}{8}\cos 4x$	No
$\boldsymbol{\frac{1}{x}}$		$\ln x$	No
$\boldsymbol{\ln x}$	Use IBP, where $u = \ln x, \frac{dv}{dx} = \ln x$	$x \ln x - x$	No
$\boldsymbol{\frac{x}{x+1}}$	Use algebraic division. $\frac{x}{x+1} \equiv 1 - \frac{1}{x+1}$	$x - \ln\lvert x+1\rvert$	
$\boldsymbol{\frac{1}{x(x+1)}}$	Use partial fractions.	$\ln\lvert x\rvert - \ln\lvert x+1\rvert$	

Summary of Key Points

$f(x)$	How to deal with it	$\int f(x)\,dx$ (+constant)
$\dfrac{4x}{x^2+1}$	Reverse chain rule. Of form $\int \frac{kf'(x)}{f(x)}$	$2\ln\lvert x^2+1\rvert$
$\dfrac{x}{(x^2+1)^2}$	Power around denominator so NOT of form $\int \frac{kf'(x)}{f(x)}$. Rewrite as product. $x(x^2+1)^{-2}$ Reverse chain rule (i.e. "Consider $y = (x^2+1)^{-1}$" and differentiate)	$-\frac{1}{2}(x^2+1)^{-1}$
e^{2x+1} $\dfrac{1}{1-3x}$	For any function where 'inner function' is linear expression, divide by coefficient of x	$\frac{1}{2}e^{2x+1}$ $-\frac{1}{3}\ln\lvert 1-3x\rvert$
$x\sqrt{2x+1}$	IBP or use sensible substitution. $u = 2x+1$ or even better, $u^2 = 2x+1$.	$\frac{1}{15}(2x+1)^{\frac{3}{2}}(3x-1)$
$\sin^5 x\ \cos x$	Reverse chain rule.	$\frac{1}{6}\sin^6 x$