KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 13
Pure Mathematics
8 Parametric equations

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

Prior knowledge check

Prior knowledge check

1 Rearrange to make t the subject:

a $x = 4t - kt$ **b** $y = 3t^2$ **c** $y = 2 - 4\ln t$ **d** $x = 1 + 2e^{-3t}$

← **GCSE Mathematics; Year 1, Chapter 14**

2 Write in terms of powers of $\cos x$:

a $4 + 3\sin^2 x$ **b** $\sin 2x$

c $\cot x$ **d** $2\cos x + \cos 2x$

← **Section 7.2**

3 State the ranges of the following functions.

a $y = \ln(x + 1), x > 0$ **b** $y = 2\sin x, 0 < x < \pi$

c $y = x^2 + 4x - 2, -4 < x < 1$ **d** $y = \dfrac{1}{2x + 5}, x > -2$

← **Section 2.2**

4 A circle has centre (0, 4) and radius 5. Find the coordinates of the points of intersection of the circle and the line with equation $2y - x - 10 = 0$.

← **Year 1, Chapter 6**

8.1) Parametric equations

Cartesian equations and co-ordinates – these are what you have largely met to this point. They typically are written as $y = f(x)$. The x y axis and (x, y) co-ordinates are all part of a system referred to as cartesian axes, co-ordinates, equations, etc.

Examples:

$$y = 3cosx - e^{3x} + lnx$$
$$x^2 + (y-1)^2 = 64$$

Parametric equations are equations where the x and y co-ordinates are each defined by functions of some other *parameter* (variable).

Examples:

$$\mathrm{x} = 3t\ , \mathrm{y} = 2e^{3t}$$
$$\mathrm{x} = 2cos\theta\ , \mathrm{y} = 2sin\theta$$

If $x = p(t)$ and $y = q(t)$ can be written as $y = f(x)$, then the domain of f is the range of p...

and the range of f is the range of q.

Notes

Worked Example

A curve has parametric equations

$$x = \ln(t+5), \qquad y = \frac{1}{t+7}, \qquad t > -4$$

Find:

a) A Cartesian equation of the curve in the form $y = f(x)$

b) The domain and range of $f(x)$

Worked Example

A curve has parametric equations

$$x = \ln t\,, \qquad y = t^3 - 4, \qquad t > 0$$

Find:

a) A Cartesian equation of the curve in the form $y = f(x)$

b) The domain and range of $f(x)$

Worked Example

A curve has parametric equations

$$x = \frac{3t}{1-t}, \qquad y = 5t + \frac{2}{t},$$

Show that the Cartesian equation of the curve is

$$y = \frac{ax^2 + bx + c}{x(x+3)}$$

where a, b and c are constants to be found.

T.200: 8A all Qs, P.60: all Qs

13. A curve C has parametric equations

$$x = \frac{t^2 + 5}{t^2 + 1} \qquad y = \frac{4t}{t^2 + 1} \qquad t \in \mathbb{R}$$

Show that all points on C satisfy

$$(x - 3)^2 + y^2 = 4$$

(3)

Your Turn

13. A curve C has parametric equations

$$x = \frac{t^2 + 9}{t^2 + 3} \qquad y = \frac{6\sqrt{3}t}{t^2 + 3} \qquad t \in \mathbb{R}$$

Show that all points on C satisfy

$$9(x-2)^2 + y^2 = 9$$

(3)

(Total for Question 13 is 3 marks)

8.2) Using trigonometric identities

Fundamental:

- $\sin^2 x + \cos^2 x = 1$
- $\sin^2 x = 1 - \cos^2 x$
- $\cos^2 x = 1 - \sin^2 x$

Derived:

- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \csc^2 x$

2. Reciprocal Identities

- $\csc x = \frac{1}{\sin x}$
- $\sec x = \frac{1}{\cos x}$
- $\cot x = \frac{1}{\tan x}$

Addition:

- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
- $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

5. Double Angle Identities

- $\sin 2x = 2\sin x \cos x$

Cosine (3 forms!):

- $\cos 2x = \cos^2 x - \sin^2 x$
- $\cos 2x = 2\cos^2 x - 1$
- $\cos 2x = 1 - 2\sin^2 x$

Tangent:

- $\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$

Notes

Worked Example

A curve has parametric equations

$$x = \cos t\,, \qquad y = \sin 2t\,, \qquad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

Find:

a) A Cartesian equation of the curve in the form $y = f(x)$

b) The valid domain and range of $f(x)$

Worked Example

A curve has parametric equations

$$x = 4\cos t\,, \qquad y = \cos 2t - 1, \qquad 0 \leq t \leq \pi$$

Find a Cartesian equation of the curve in the form
$y = f(x), -k \leq x \leq k$, stating the value of the constant k

Worked Example

A curve has parametric equations

$$x = \cot t + 1, \qquad y = cosec^2\, t - 3, \qquad 0 < t < \pi$$

Find a Cartesian equation of the curve in the form $y = f(x)$ and state the domain of x for which the curve is defined

Worked Example

A curve has parametric equations

$$x = \sqrt{5}\sin 2t\,, \qquad y = 10\sin^2 t\,, \qquad 0 \le t < \pi$$

Find a Cartesian equation of the curve

T.204: 8B Qs 1-4, P.61: Qs 1-5

Worked Example

A curve has parametric equations

$$x = 2\sin t\,, \qquad y = \sin\left(t + \frac{\pi}{6}\right), \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

Find a Cartesian equation of the curve in the form $y = f(x)$ and state the domain of x for which the curve is defined

Worked Example

A curve has parametric equations

$$x = \tan t\,, \qquad y = 5\sin(t - \pi)\,, \qquad 0 < t < \frac{\pi}{2}$$

Find a Cartesian equation of the curve

T.204: 8B Qs 5+, P.62: Qs 6,7

4.

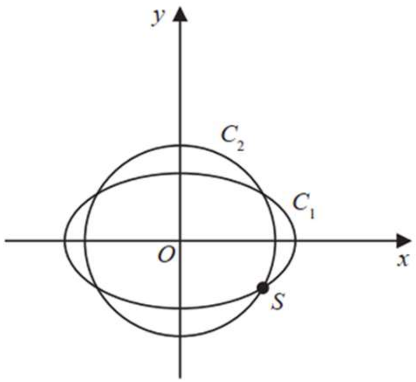


Figure 2

The curve C_1 with parametric equations

$$x = 10\cos t, \quad y = 4\sqrt{2}\sin t, \quad 0 \leqslant t < 2\pi$$

meets the circle C_2 with equation

$$x^2 + y^2 = 66$$

at four distinct points as shown in Figure 2.

Given that one of these points, S, lies in the 4th quadrant, find the Cartesian coordinates of S.

(6)

4.

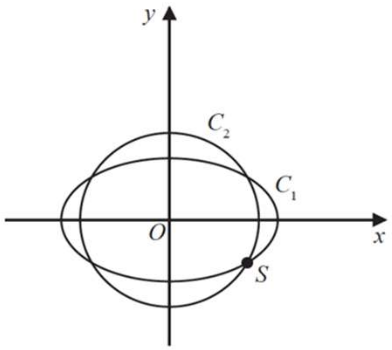


Figure 2

The curve C_1 with parametric equations

$$x = 22\cos t, \quad y = 2\sqrt{5}\sin t, \qquad 0 \leq t < 2\pi,$$

meets the circle C_2 with equation

$$x^2 + y^2 = 130$$

at four distinct points as shown in Figure 2.

Given that one of these points, S, lies in the 4th quadrant, find the Cartesian coordinates of S to 3 significant figures.

(Total for Question 4 is 6 marks)

8.3) Curve sketching

Make tables such as:

t	-2	-1	0	1	2	3
x						
y						

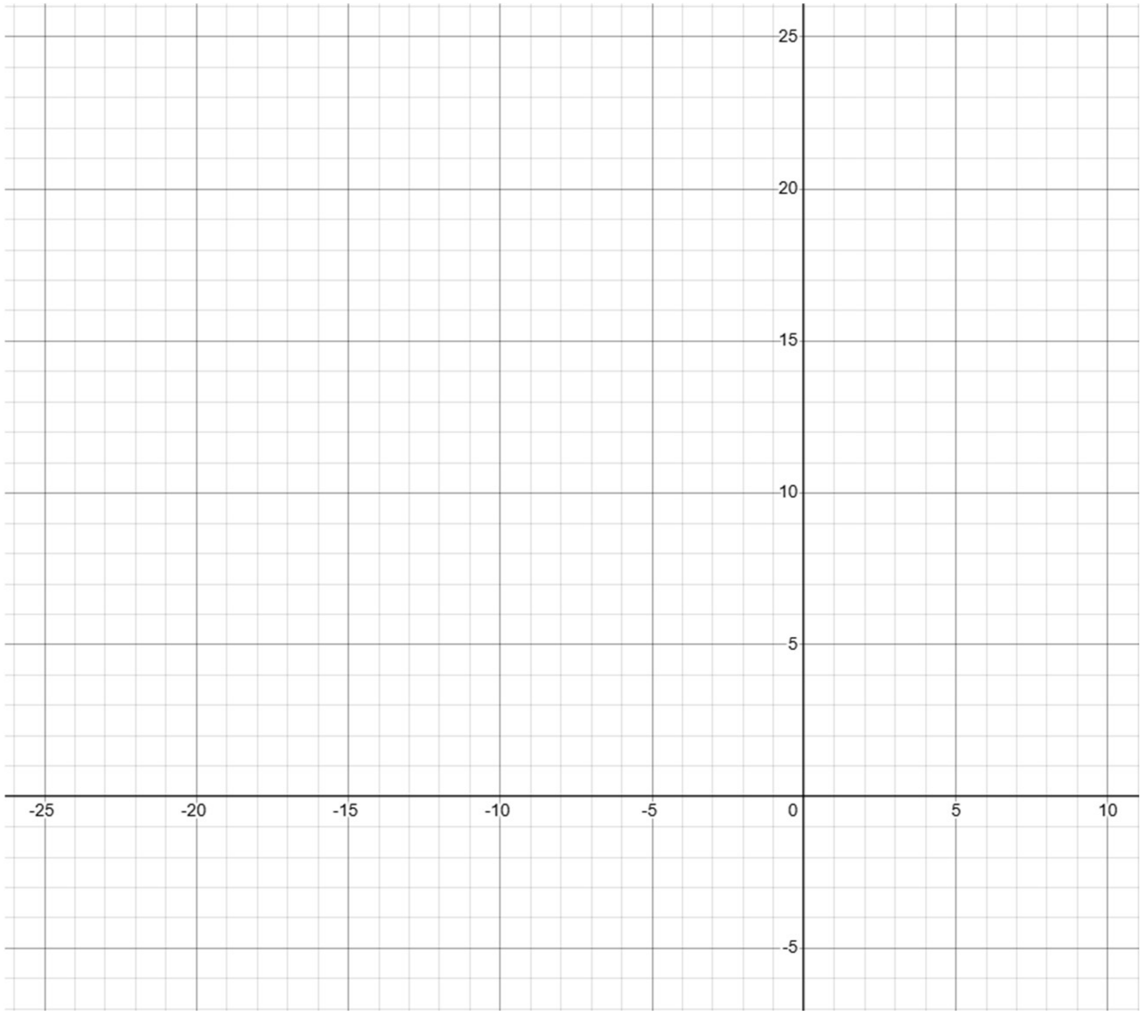
Then plot points (x, y)

Worked Example

Draw the curve given by the parametric equations

$$x = 3t, \qquad y = t^2, \qquad -5 \le t \le 1$$

t	-5	-4	-3	-2	-1	0	1
x							
y							

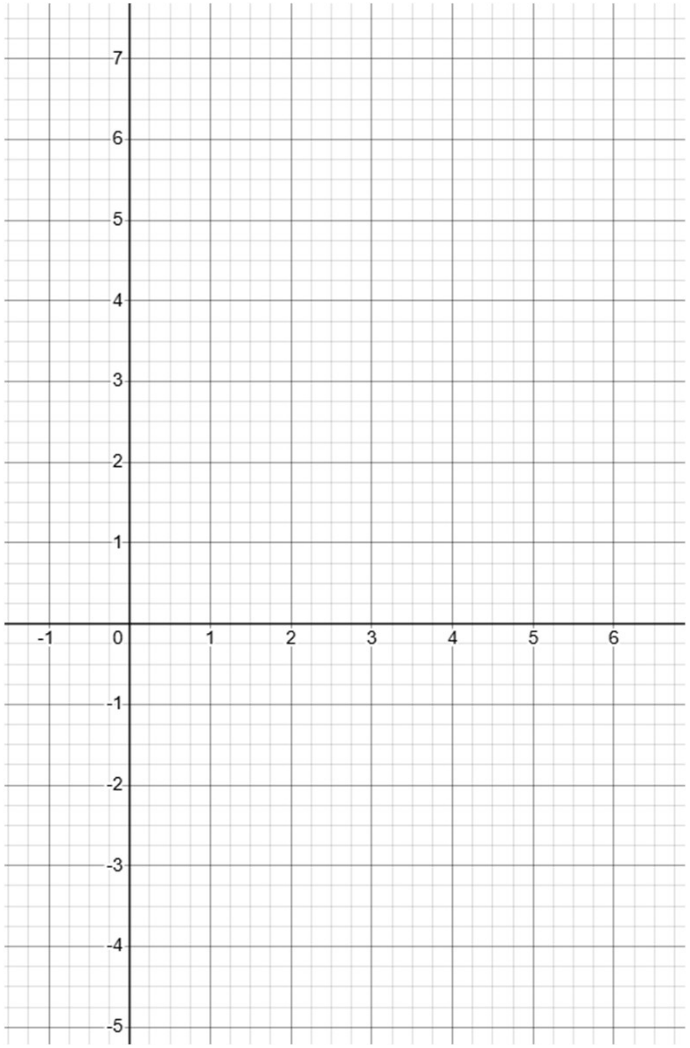


Worked Example

Draw the curve given by the parametric equations

$$x = 2 - t, \qquad y = t^2 - 3, \qquad -3 \le t \le 2$$

t	-3	-2	-1	0	1	2
x						
y						



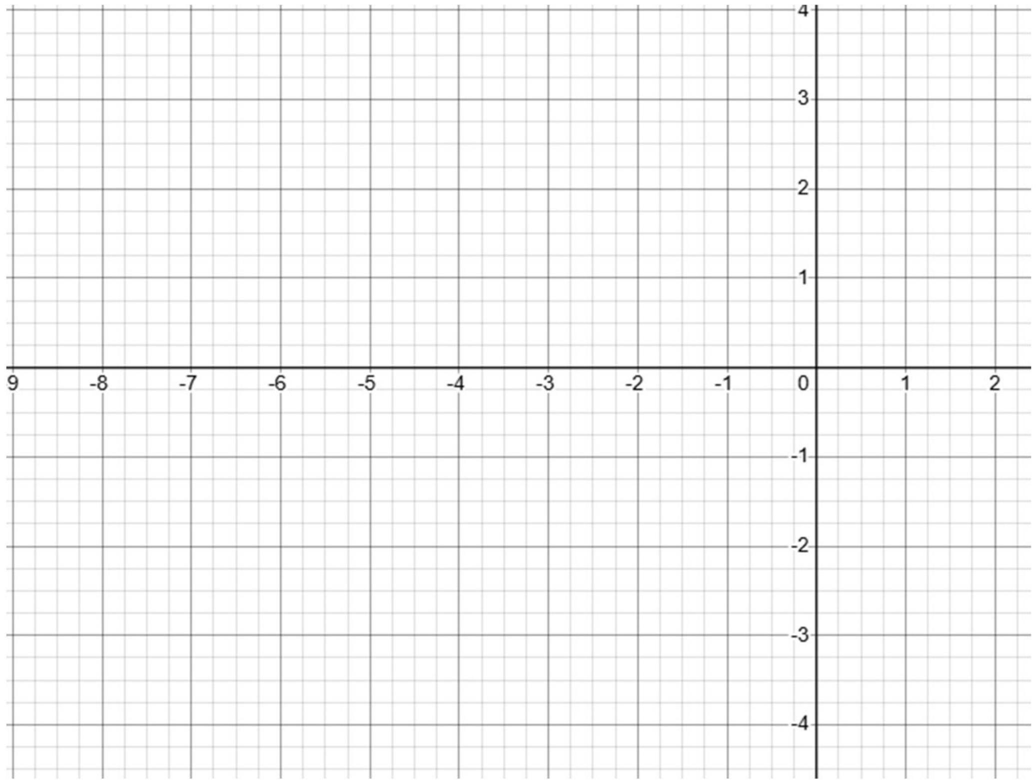
T.207: 8C Qs 1,2 P.63: Q 1

Worked Example

Draw the curve given by the parametric equations

$$x = 2\cos t - 3, \qquad y = 4\sin t\,, \qquad 0 \le t \le 2\pi$$

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
x									
y									



T.207: 8C Qs 3+ P.63: Q 2+

8.4) Points of intersection

We can find where a parametric curve crosses a particular axis or where curves cross each other.

The key is to first find the value of the parameter t.

Worked Example

A curve C is given by the parametric equations

$$x = t^2,\ y = 2t,\ \ t \in \mathbb{R}$$

Find the coordinates of the point(s) of intersection between the curve C and the line $x + y - 8 = 0$

T.211: 8D Qs 1-7,13 P.63: Qs 1,3,4

Worked Example

A curve C is given by the parametric equations

$$x = \cos t - \sin t\,, \qquad y = \left(t + \frac{\pi}{6}\right)^2, \qquad -\frac{\pi}{3} < t < \frac{3\pi}{2}$$

a) Find the point where the curve intersects the line $y = \pi^2$.

b) Find the coordinates of the points where the curve cuts the y-axis.

T.211: 8D Qs 8,12,14,16 P.63: Qs 6,7

Worked Example

A curve C is given by the parametric equations

$$x = 1 - \frac{1}{3}t, \qquad y = 3^t - 1, \qquad t \in \mathbb{R}$$

Find the coordinates of the x and y intercepts

Worked Example

A curve C is given by the parametric equations

$$x = e^{3t}, \qquad y = e^t + 1, \qquad t \in \mathbb{R}$$

A straight line l passes through the points A and B where $t = \ln 3$ and $t = \ln 4$ respectively.
Find an equation for l in the form $ax + by + c = 0$

T.211: 8D Qs 9,15,17 P.63: Qs 2,5,8

14. A curve C has parametric equations

$$x = 3 + 2\sin t, \qquad y = 4 + 2\cos 2t, \qquad 0 \leqslant t < 2\pi$$

(a) Show that all points on C satisfy $y = 6 - (x - 3)^2$

(2)

(b) (i) Sketch the curve C.

(ii) Explain briefly why C does not include all points of $y = 6 - (x - 3)^2, \quad x \in \mathbb{R}$

(3)

The line with equation $x + y = k$, where k is a constant, intersects C at two distinct points.

(c) State the range of values of k, writing your answer in set notation.

(5)

8.5) Modelling with parametric equations

Worked Example

A plane's position at time t seconds after take-off can be modelled with the following parametric equations:

$$x = (v\cos\theta)t \text{ m}, \qquad y = (v\sin\theta)t \text{ m}, \qquad t > 0$$

where v is the speed of the plane, θ is the angle of elevation of its path, x is the horizontal distance travelled and y is the vertical distance travelled, relative to a fixed origin.

When the plane has travelled 500m horizontally, it has climbed 125m.

Given that the plane's speed is 40 m s^{-1}

a) find the parametric equations for the plane's motion.

b) find the vertical height of the plane after 20 seconds.

c) show that the plane's motion is a straight line.

d) explain why the domain of t, $t > 0$, is not realistic.

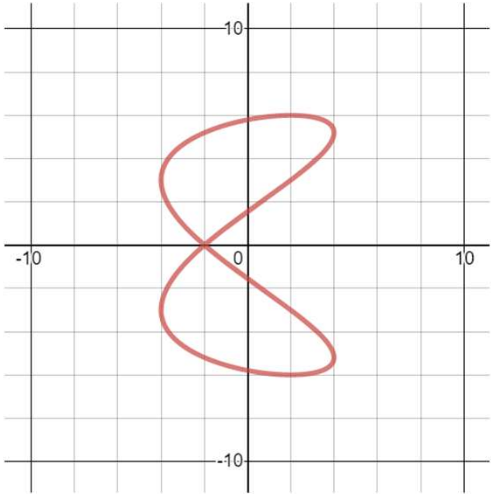
Worked Example

The motion of a figure skater relative to a fixed origin, O, at time t minutes is modelled using the parametric equations

$$x = 4\cos 10t, \qquad y = 6\sin\left(5t - \frac{\pi}{3}\right), \qquad t \geq 0$$

where x and y are measured in metres.

a) Find the coordinates of the figure skater at the beginning of his motion.
b) Find the coordinates of the point where the figure skater intersects his own path.
c) Find the coordinates of the points where the path of the figure skater crosses the y-axis.
d) Determine how long it takes the figure skater to complete one complete figure-of-eight motion.



T.218: 8E Qs 1-4 P.65: Qs 1-4

Worked Example

A stone is thrown from the top of a $50\ m$ high cliff with an initial speed of $5\ ms^{-1}$ at an angle of $30°$ above the horizontal. Its position after t seconds can be described using the parametric equations

$$x = \frac{5\sqrt{3}}{2} t\ m, \qquad y = \left(-4.9t^2 + \frac{5\sqrt{3}}{2} t + 50\right) m, \qquad 0 \le t \le k$$

where x is the horizontal distance, y is the vertical distance from the ground and k is a constant.
Given that the model is valid from the time the stone is thrown to the time it hits the ground,

a) find the value of k
b) find the horizontal distance travelled by the stone once it hits the ground

T.218: 8E Qs 5+ P.65: Qs 5+

Figure 6 shows a sketch of the curve C with parametric equations

$$x = 2\tan t + 1 \qquad y = 2\sec^2 t + 3 \qquad -\frac{\pi}{4} \leqslant t \leqslant \frac{\pi}{3}$$

The line l is the normal to C at the point P where $t = \frac{\pi}{4}$

~~(a) Using parametric differentiation, show that an equation for l is~~

$$y = -\frac{1}{2}x + \frac{17}{2}$$

(5)

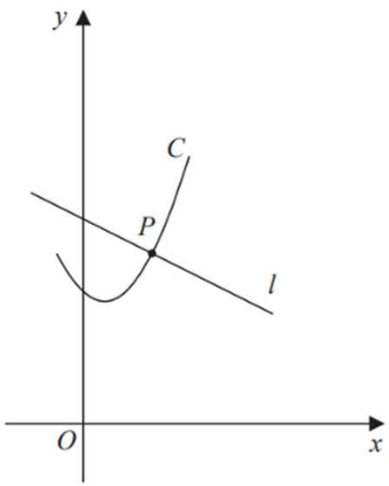


Figure 6

(b) Show that all points on C satisfy the equation

$$y = \frac{1}{2}(x-1)^2 + 5$$

(2)

The straight line with equation

$$y = -\frac{1}{2}x + k \qquad \text{where } k \text{ is a constant}$$

intersects C at two distinct points.

(c) Find the range of possible values for k.

(5)

Your Turn

16.

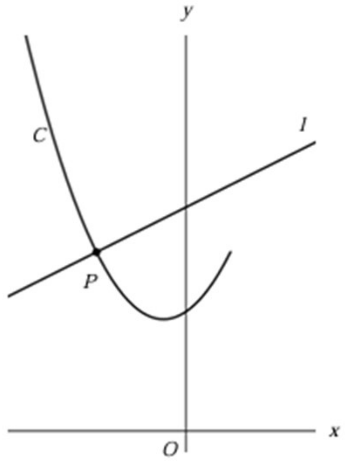


Figure 6

Figure 6 shows a sketch of the curve C with parametric equations

$$x = 3\cot t - 1 \qquad y = 3\,\mathrm{cosec}^2 t + 2 \qquad \frac{\pi}{4} \leq t < \theta$$

The line l is the normal to C at the point P where $t = \frac{3\pi}{4}$

(a) Using parametric differentiation, show that an equation for l is

$$y = \frac{1}{2}x + 10$$

(5)

(b) Show that all points on C satisfy the equation

$$y = \frac{1}{3}(x+1)^2 + 5$$

(2)

The straight line with equation

$$y = \frac{1}{2}x + k \quad \text{where } k \text{ is a constant}$$

intersects C at two distinct points.

(c) Find the range of possible values for k.

(5)

(Total for Question 16 is 12 marks)

Extract from Formulae book

Trigonometric identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad (A \pm B \neq (k + \tfrac{1}{2})\pi)$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Logarithms and exponentials

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$e^{x \ln a} = a^x$$

A2 SAMs Paper 1

Parametric Equations

5. A curve C has parametric equations

$$x = 2t - 1, \qquad y = 4t - 7 + \frac{3}{t}, \qquad t \neq 0$$

Show that the Cartesian equation of the curve C can be written in the form

$$y = \frac{2x^2 + ax + b}{x + 1}, \qquad x \neq -1$$

where a and b are integers to be found.

(3)

Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

Question	Scheme	Marks	AOs
5	Attempts to substitute $t = \frac{x+1}{2}$ into $y \Rightarrow y = 4\left(\frac{x+1}{2}\right) - 7 + \frac{6}{(x+1)}$	M1	2.1
	Attempts to write as a single fraction $y = \frac{(2x-5)(x+1)+6}{(x+1)}$	M1	2.1
	$y = \frac{2x^2 - 3x + 1}{x+1}$ $\quad a = -3, b = 1$	A1	1.1b
			(3 marks)

Summary of Key Points

Summary of key points

1. A curve can be defined using parametric equations $x = \mathrm{p}(t)$ and $y = \mathrm{q}(t)$. Each value of the parameter, t, defines a point on the curve with coordinates $(\mathrm{p}(t), \mathrm{q}(t))$.

2. You can convert between parametric equations and Cartesian equations by using substitution to eliminate the parameter.

3. For parametric equations $x = \mathrm{p}(t)$ and $y = \mathrm{q}(t)$ with Cartesian equation $y = \mathrm{f}(x)$:
   - the domain of $\mathrm{f}(x)$ is the range of $\mathrm{p}(t)$
   - the range of $\mathrm{f}(x)$ is the range of $\mathrm{q}(t)$

4. You can use parametric equations to model real-life situations. In mechanics you will use parametric equations with time as a parameter to model motion in two dimensions.

T.220: mixed ex ,P67: BSG Qs