KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 13
Pure Mathematics
P2 6 Trigonometric Functions

HGS Maths

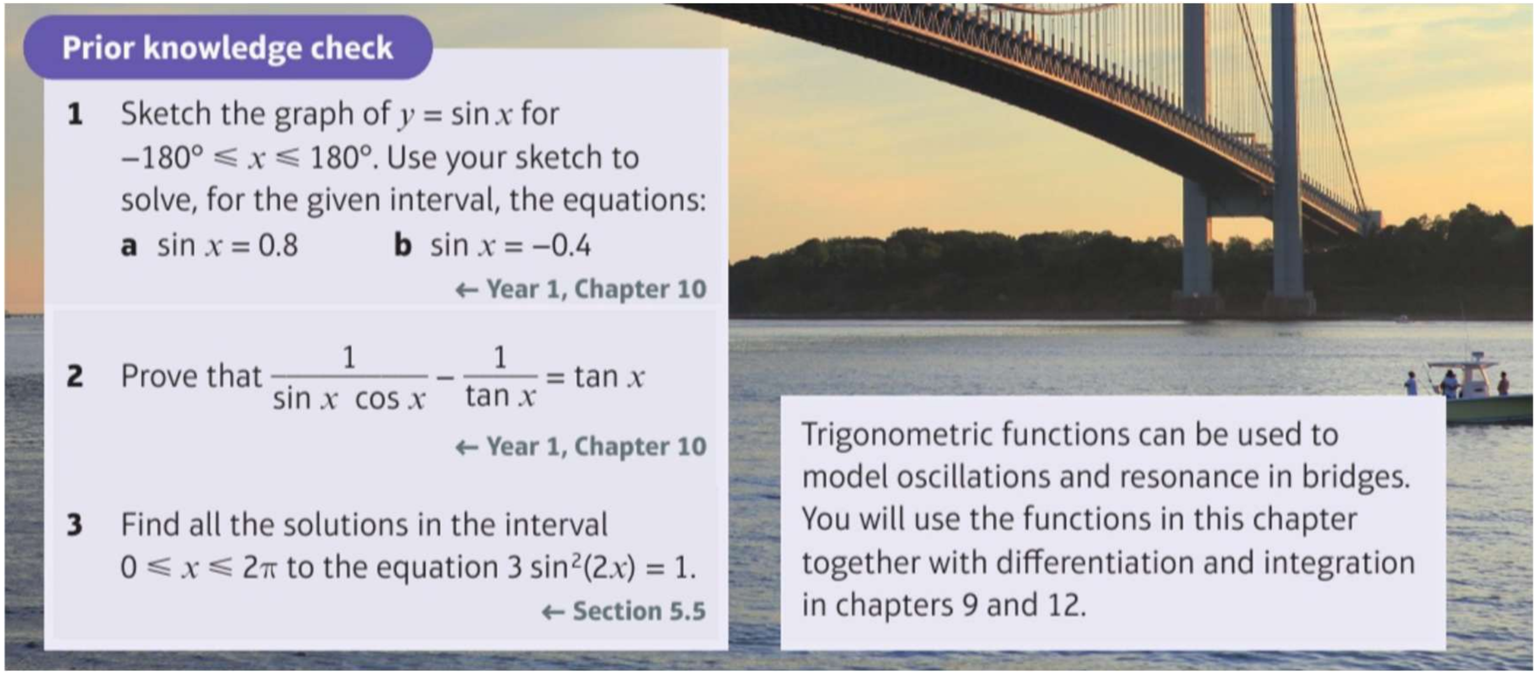
Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

Prior knowledge check



Prior knowledge check

1 Sketch the graph of $y = \sin x$ for $-180° \leqslant x \leqslant 180°$. Use your sketch to solve, for the given interval, the equations:

a $\sin x = 0.8$ **b** $\sin x = -0.4$

← **Year 1, Chapter 10**

2 Prove that $\dfrac{1}{\sin x \ \cos x} - \dfrac{1}{\tan x} = \tan x$

← **Year 1, Chapter 10**

3 Find all the solutions in the interval $0 \leqslant x \leqslant 2\pi$ to the equation $3\sin^2(2x) = 1$.

← **Section 5.5**

Trigonometric functions can be used to model oscillations and resonance in bridges. You will use the functions in this chapter together with differentiation and integration in chapters 9 and 12.

Reciprocal Trigonometric Functions

( ) brackets not *required* here but can be used

$$\sec(x) = \frac{1}{\cos(x)}$$

Short for "**secant**"

$$\text{cosec}(x) = \frac{1}{\sin(x)}$$

Short for "**cosecant**"

$$\cot(x) = \frac{1}{\tan(x)} \; or \; \frac{\cos(x)}{\sin(x)}$$

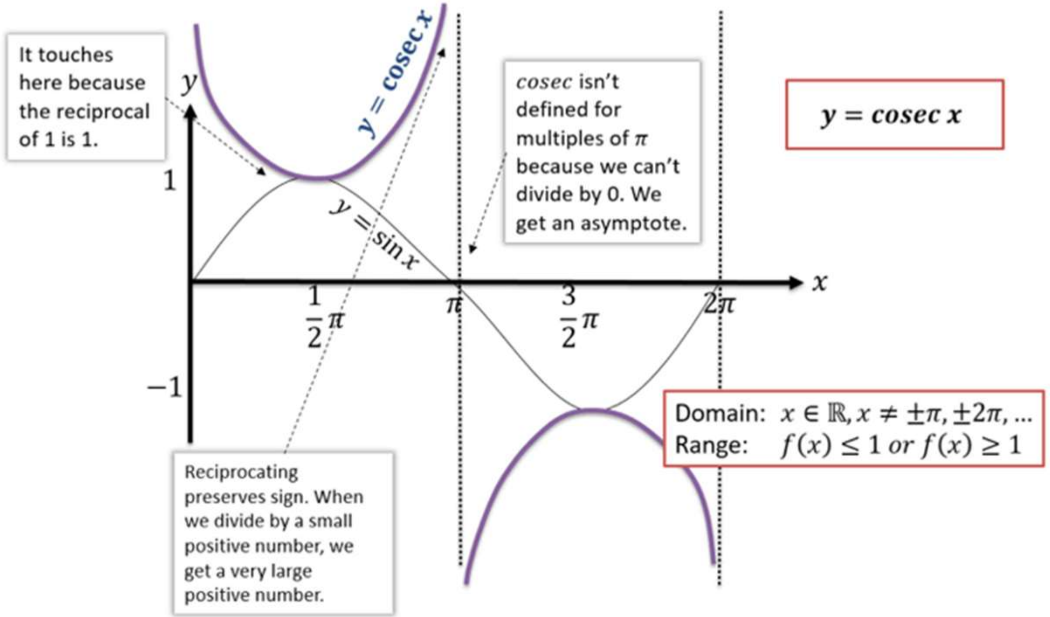
Short for "**cotangent**"

We typically use this version instead of $\frac{1}{\tan x}$ when doing proof questions.

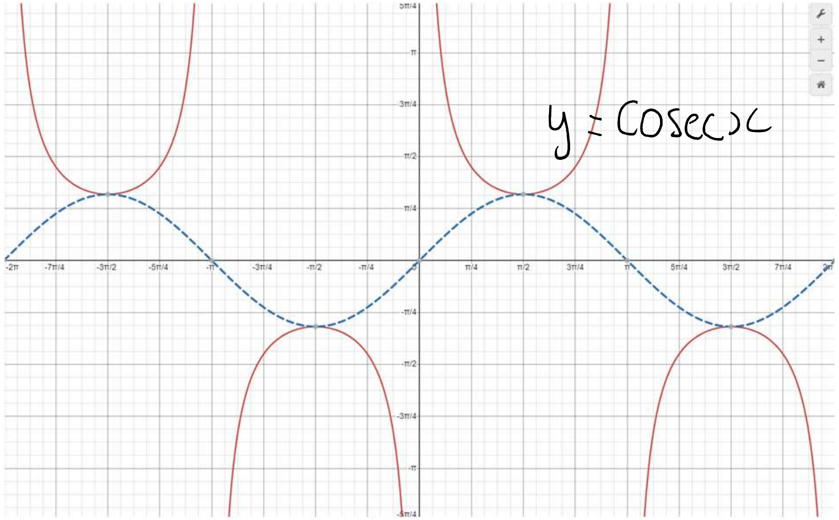
Tip: To remember these, look at the **3rd letter**: sec's 3rd is 'c' so it's 1 over **cos**.

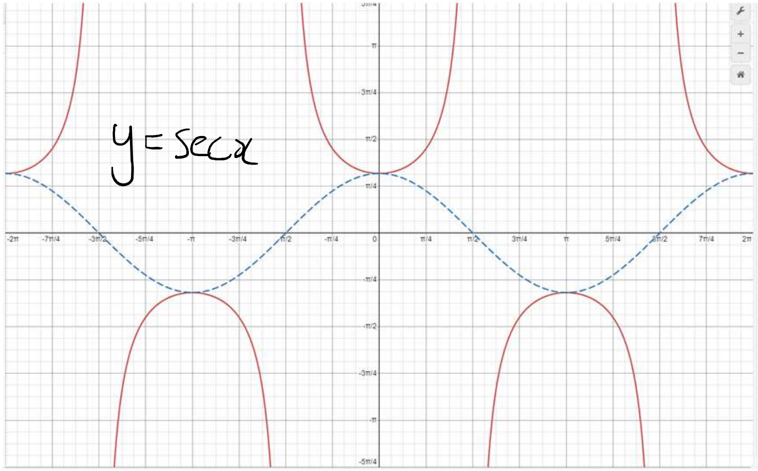
Sketches

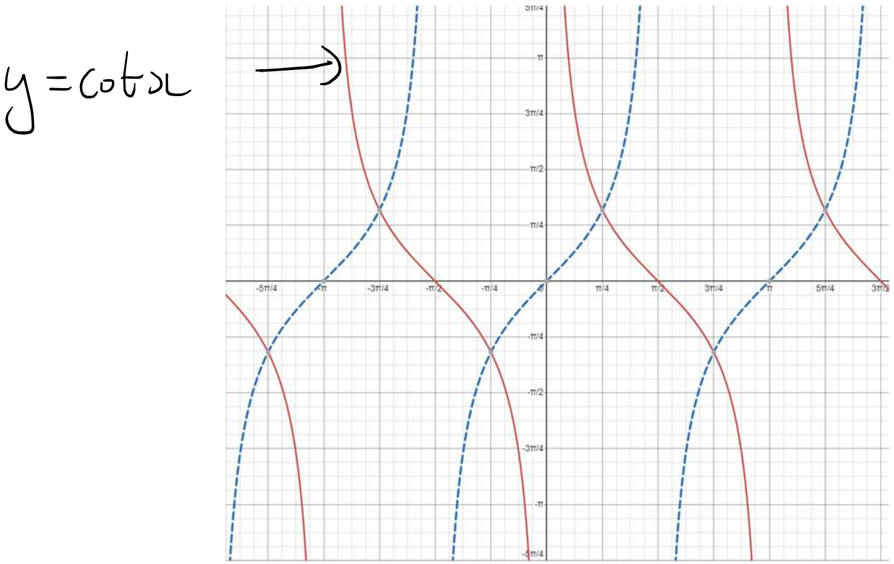
To draw a graph of $y = cosec\ x$, start with a graph of $y = \sin x$, then consider what happens when we reciprocate each y value.



Graphs



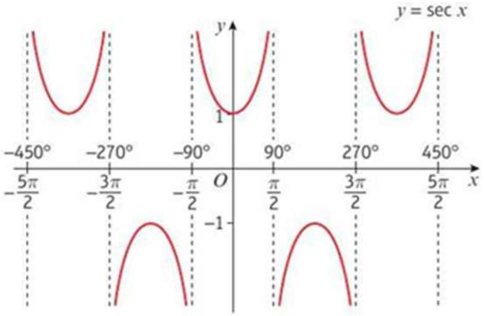




Notes

6.2) Graphs of $\sec x$, $\operatorname{cosec} x$ and $\cot x$

- The graph of $y = \sec x$, $x \in \mathbb{R}$, has symmetry in the y-axis and has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\cos x = 0$.

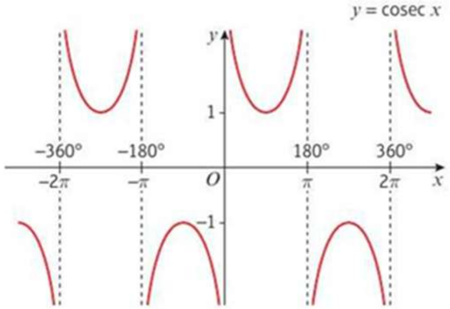


Notation The domain can also be given as $x \in \mathbb{R}, x \neq \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}$

$\mathbb{Z}$ is the symbol used for **integers**, i.e. positive and negative whole numbers including 0.

- The domain of $y = \sec x$ is $x \in \mathbb{R}$, $x \neq 90°, 270°, 450°, \ldots$ or any odd multiple of 90°
- In radians the domain is $x \in \mathbb{R}$, $x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ or any odd multiple of $\frac{\pi}{2}$
- The range of $y = \sec x$ is $y \leqslant -1$ or $y \geqslant 1$

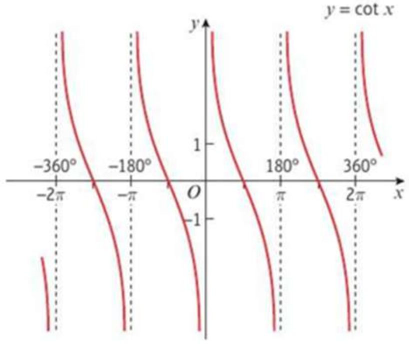
- The graph of $y = \operatorname{cosec} x$, $x \in \mathbb{R}$, has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\sin x = 0$.



Notation The domain can also be given as $x \in \mathbb{R}, x \neq n\pi, n \in \mathbb{Z}$.

- The domain of $y = \operatorname{cosec} x$ is $x \in \mathbb{R}$, $x \neq 0°, 180°, 360°, \ldots$ or any multiple of 180°
- In radians the domain is $x \in \mathbb{R}$, $x \neq 0, \pi, 2\pi, \ldots$ or any multiple of π
- The range of $y = \operatorname{cosec} x$ is $y \leqslant -1$ or $y \geqslant 1$

- The graph of $y = \cot x$, $x \in \mathbb{R}$, has period 180° or π radians. It has vertical asymptotes at all the values of x for which $\tan x = 0$.



- The domain of $y = \cot x$ is $x \in \mathbb{R}$, $x \neq 0°, 180°, 360°, \ldots$ or any multiple of 180°
- In radians the domain is $x \in \mathbb{R}$, $x \neq 0, \pi, 2\pi, \ldots$ or any multiple of π
- The range of $y = \cot x$ is $y \in \mathbb{R}$

Notation The domain can also be given as $x \in \mathbb{R}, x \neq n\pi, n \in \mathbb{Z}$.

Worked Example

Sketch the graph in the interval $-2\pi \leq x \leq 2\pi$:

$$y = cosec\left(x + \frac{\pi}{4}\right)$$

$$y = \cot(x - \frac{\pi}{3})$$

T.148 6B: Qs 2-6, P.43 6.2: Qs2-5

Worked Example

Sketch the graph in the interval $-2\pi \leq x \leq 2\pi$:

$y = 2cosec\ x$

$y = 3\cot x$

Worked Example

Sketch the graph in the interval $0 \leq x \leq 2\pi$:

$y = 6 + cosec\ 4x$

$y = \cot 3x - 5$

Worked Example

State the range of:

$y = cosec\ x, x \in \mathbb{R}, x \neq n\pi, n \in \mathbb{Z}$

$y = cot\ x, x \in \mathbb{R}, x \neq n\pi, n \in \mathbb{Z}$

T.148 6B: Qs 7-8

Worked Example

Find the range of values of k for which $2 + 7\sec x = k$ has no solutions.

Find the range of values of k for which $3\ cosec\ x\ - 5 = k$ has no solutions

Worked Example

Find the maximum and minimum of the graph, stating the smallest positive values of θ at which they occur:

$$y = \frac{1}{2 + 3\sec\theta}$$

T.148 6B: Qs -+, P.43 6.2: Qs6+

6.3) Using $\sec x$, $\operatorname{cosec} x$ and $\cot x$

Using sec, cosec, cot - proving identities

Questions in the exam usually come in two types: (a) 'prove' questions requiring to prove some identity and (b) 'solve' questions.

Tip 1: Get everything in terms of sin and cos first (using $\cot x = \frac{\cos x}{\sin x}$ rather than $\cot x = \frac{1}{\tan x}$)

Tip 2: Whenever you have algebraic fractions being added/subtracted, whether $\frac{a}{b} + \frac{c}{d}$ or $\frac{a}{b} + c$, combine them into one (as we can typically then use $\sin^2 x + \cos^2 x = 1$)

Worked Example

Simplify:

$$\sin\theta \cos\theta \ \left(cosec\ \theta - \sec\theta\right)$$

Simplify:

$$cos\ \theta \tan\theta \operatorname{cosec}\theta$$

T.152 6C: Qs 1-3, P.45 6.3: Qs 1-2

Worked Example

Prove that:

$$\frac{\tan\theta \;\; sec\;\theta}{\sec^2\theta + cosec^2\;\theta} \equiv \sin^3\theta$$

Worked Example

Prove that:

$$\operatorname{cosec} x - \sin x \equiv \cos x \cot x$$

Worked Example

Prove that:

$$(1 + \sin x)(sec\, x - \tan x) \equiv \cos x$$

T.152 6C: Q 4, P.45 6.3: Qs 5-6

Worked Example

Solve in the interval $0 \leq \theta \leq 2\pi$:

$$cosec\, \theta = 1$$

Solve in the interval $0 \leq \theta \leq 2\pi$:

$$\cot\theta = \sqrt{3}$$

Worked Example

Solve in the interval $0 \leq \theta \leq 2\pi$:

$$cosec\ \theta = 2$$

Solve in the interval $0 \leq \theta \leq 2\pi$:

$$\cot \theta = -3$$

Worked Example

Solve in the interval $0 \leq \theta \leq 2\pi$:

$$cosec\,\theta = 0$$

Solve in the interval $0 \leq \theta \leq 2\pi$:

$$\cot\theta = 0$$

Worked Example

Solve in the interval $-180° \leq \theta \leq 180°$:

$$cosec\ \theta = -\sqrt{2}$$

Solve in the interval $-180° \leq \theta \leq 180°$:

$$\cot\theta = -\sqrt{3}$$

T.152 6C: Q 5-8, P.45 6.3: Qs 3,6

Worked Example

Solve in the interval $0° \leq \theta \leq 360°$:

$$\frac{1 - \tan x}{1 - \cot x} = 2$$

T.152 6C: Q 9+, P.45 6.3: Qs 7

12. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\operatorname{cosec}\theta - \sin\theta \equiv \cos\theta\cot\theta \qquad \theta \neq (180n)° \quad n \in \mathbb{Z}$$

(3)

(b) Hence, or otherwise, solve for $0 < x < 180°$

$$\operatorname{cosec}x - \sin x = \cos x\cot(3x - 50°)$$

(5)

Your Turn

12. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(*a*) Show that

$$\sec\theta - \cos\theta \equiv \sin\theta\tan\theta \qquad \theta \neq (180n)^\circ \qquad n \in \mathbb{Z}$$

(3)

(*b*) Hence, or otherwise, solve for $0 < x \leq 360^\circ$

$$\sec x - \cos x = \sin x \tan(2x - 30^\circ)$$

(5)

(Total for Question 12 is 8 marks)

6.4) Trigonometric identities

$$\sin^2 x + \cos^2 x = 1$$

There are just two new identities you need to know:

Dividing by $\cos^2 x$:

$$1 + \tan^2 x = \sec^2 x$$

Dividing by $\sin^2 x$:

$$1 + \cot^2 x = \operatorname{cosec}^2 x$$

Fro Tip: I used to misremember this as "$1 + \sec^2 x = \tan^2 x$". Then I imagined the Queen coming back from holiday, saying "One is tanned", i.e. the 1 goes with the $\tan^2 x$.

Fro Tip: I remember this one by starting with the above, and slapping 'co' on front of each trig function.

"Prove that $1 + \tan^2 x \equiv \sec^2 x$."

$$\sin^2 x + \cos^2 x \equiv 1$$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} \equiv \frac{1}{\cos^2 x}$$

$$\tan^2 x + 1 \equiv \sec^2 x$$

Fro Tip: This has been asked in an exam before! <u>You must explicitly show each term being divided by $\cos^2 x$.</u>

Notes

Worked Example

Prove that:

$$cosec^2\theta - \sin^2\theta \equiv \cos^2\theta\,(1 + \mathrm{cosec}^2\,\theta)$$

Worked Example

Prove that:

$$sec^4\theta - \tan^4\theta = \frac{1+\sin^2\theta}{1-\sin^2\theta}$$

Worked Example

Given that $\tan A = -\frac{3}{4}$ and angle A is obtuse, find the exact values of $\sec A$ and $\sin A$

Worked Example

Given that $\cos A = \frac{3}{5}$ and angle A is reflex, find the exact values of $\tan A$ and $\operatorname{cosec} A$

Worked Example

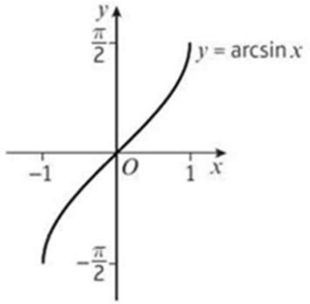
Given that $x = cosec\,\theta + \cot\theta$, express in its simplest form:

$$x^2 + \frac{1}{x^2} + 2$$

T.156 6D: Qs 7+, P.45 6.4: Qs 5+

6.5) Inverse trigonometric functions

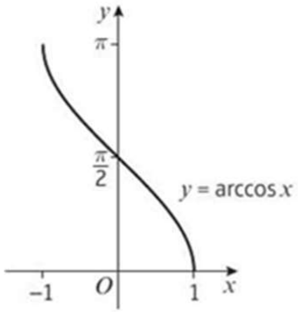
■ **The inverse function of sin x is called arcsin x.**



Hint The $\sin^{-1}$ function on your calculator will give principal values in the same range as arcsin.

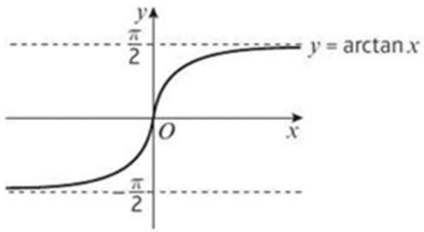
- **The domain of $y = \arcsin x$ is $-1 \leqslant x \leqslant 1$.**
- **The range of $y = \arcsin x$ is $-\frac{\pi}{2} \leqslant \arcsin x \leqslant \frac{\pi}{2}$ or $-90° \leqslant \arcsin x \leqslant 90°$.**

■ **The inverse function of cos x is called arccos x.**



- **The domain of $y = \arccos x$ is $-1 \leqslant x \leqslant 1$.**
- **The range of $y = \arccos x$ is $0 \leqslant \arccos x \leqslant \pi$ or $0° \leqslant \arccos x \leqslant 180°$.**

■ **The inverse function of tan x is called arctan x.**

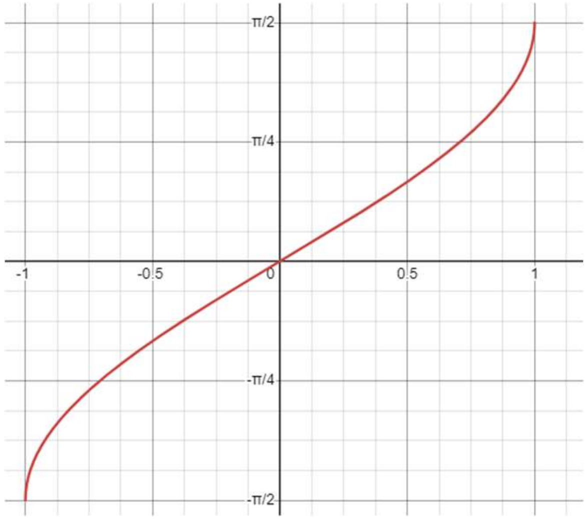


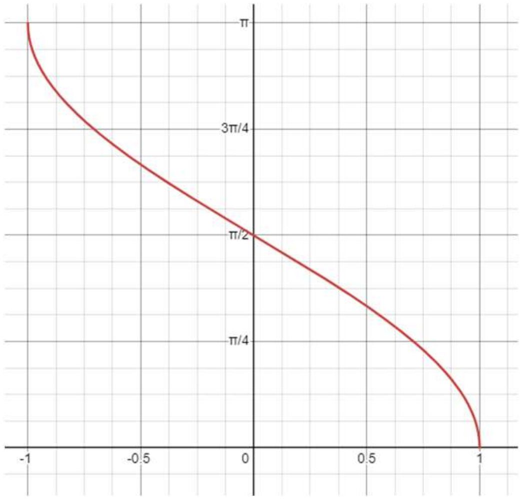
Watch out Unlike arcsin x and arccos x, the function arctan x is defined for all real values of x.

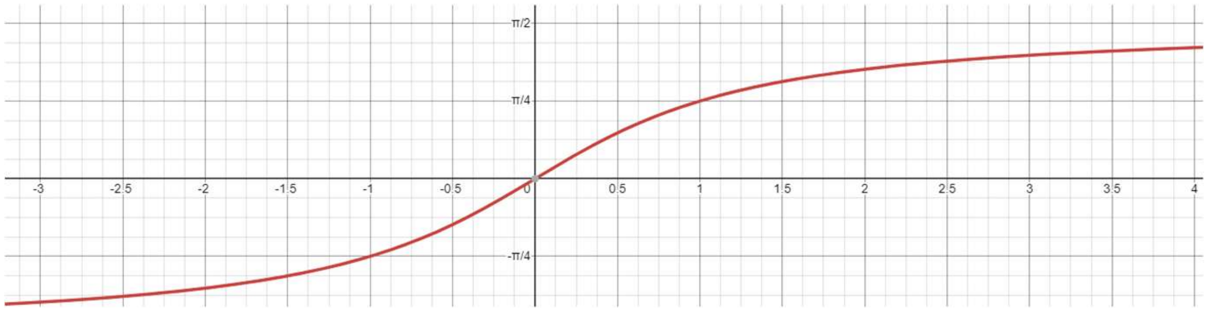
- **The domain of $y = \arctan x$ is $x \in \mathbb{R}$.**
- **The range of $y = \arctan x$ is $-\frac{\pi}{2} \leqslant \arctan x \leqslant \frac{\pi}{2}$ or $-90° \leqslant \arctan x \leqslant 90°$.**

Graphs

Fully label:







Worked Example

Work out, in radians, the values of:

a) $\arcsin\left(\frac{\sqrt{2}}{2}\right)$

b) $\arccos(1)$

c) $\arctan\left(-\sqrt{3}\right)$

T.160 6E: Qs 1-4, P.46 6.5: Qs1-4

Worked Example

Given that $y = \arcsin x$, $-1 \leq x \leq 1$ and $0 \leq y \leq \pi$,

a) Express $\arccos x$ in terms of y

b) Hence evaluate $\arcsin x - \arccos x$

Worked Example

Prove that for $0 \leq x \leq 1$, $\arcsin x = \arccos \sqrt{1 - x^2}$ and give a reason why this result is not true for $-1 \leq x \leq 0$

T.1600 6E: Qs +5, P.46 6.5: Qs5+

Extract from Formulae book

Trigonometric identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad (A \pm B \neq (k + \tfrac{1}{2})\pi)$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Small angle approximations

$$\sin\theta \approx \theta$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$

$$\tan\theta \approx \theta$$

where θ is measured in radians

Past Paper Questions

A2 2020 Paper 1

Trig – sec, cosec, cot

12. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\operatorname{cosec}\theta - \sin\theta \equiv \cos\theta\cot\theta \qquad \theta \neq (180n)^\circ \quad n \in \mathbb{Z}$$

(3)

(b) Hence, or otherwise, solve for $0 < x < 180^\circ$

$$\operatorname{cosec}x - \sin x = \cos x\cot(3x - 50^\circ)$$

(5)

Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

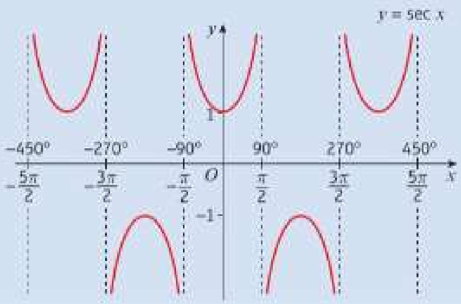
Question	Scheme	Marks	AOs
12 (a)	States or uses $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$	B1	1.2
	$\operatorname{cosec}\theta - \sin\theta = \frac{1}{\sin\theta} - \sin\theta = \frac{1-\sin^2\theta}{\sin\theta}$	M1	2.1
	$= \frac{\cos^2\theta}{\sin\theta} = \cos\theta \times \frac{\cos\theta}{\sin\theta} = \cos\theta\cot\theta$ *	A1*	2.1
		(3)	
(b)	$\operatorname{cosec}x - \sin x = \cos x\cot(3x-50^\circ)$ $\Rightarrow \cos x\cot x = \cos x\cot(3x-50^\circ)$		
	$\cot x = \cot(3x-50^\circ) \Rightarrow x = 3x-50^\circ$	M1	3.1a
	$x = 25^\circ$	A1	1.1b
	Also $\cot x = \cot(3x-50^\circ) \Rightarrow x+180^\circ = 3x-50^\circ$	M1	2.1
	$x = 115^\circ$	A1	1.1b
	Deduces $x = 90^\circ$	B1	2.2a
		(5)	
			(8 marks)

Summary of Key Points

Summary of key points

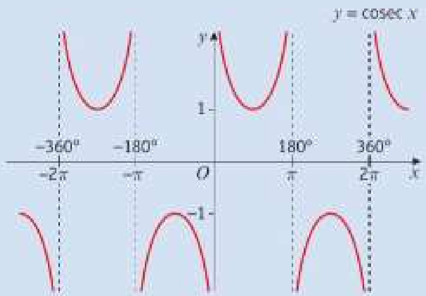
1
- $\sec x = \dfrac{1}{\cos x}$ (undefined for values of x for which $\cos x = 0$)
- $\operatorname{cosec} x = \dfrac{1}{\sin x}$ (undefined for values of x for which $\sin x = 0$)
- $\cot x = \dfrac{1}{\tan x}$ (undefined for values of x for which $\tan x = 0$)
- $\cot x = \dfrac{\cos x}{\sin x}$

2 The graph of $y = \sec x$, $x \in \mathbb{R}$, has symmetry in the y-axis and has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\cos x = 0$.



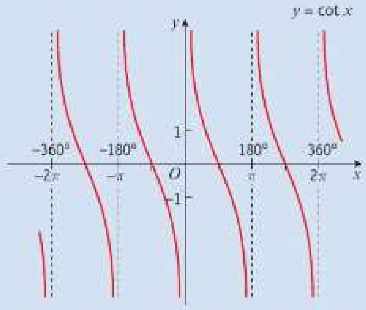
- The domain of $y = \sec x$ is $x \in \mathbb{R}$, $x \neq 90°, 270°, 450°, \ldots$ or any odd multiple of 90°.
- In radians the domain is $x \in \mathbb{R}$, $x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ or any odd multiple of $\frac{\pi}{2}$
- The range of $y = \sec x$ is $y \leqslant -1$ or $y \geqslant 1$.

3 The graph of $y = \operatorname{cosec} x$, $x \in \mathbb{R}$, has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\sin x = 0$.



- The domain of $y = \operatorname{cosec} x$ is $x \in \mathbb{R}$, $x \neq 0°, 180°, 360°, \ldots$ or any multiple of 180°.
- In radians the domain is $x \in \mathbb{R}$, $x \neq 0, \pi, 2\pi, \ldots$ or any multiple of π
- The range of $y = \operatorname{cosec} x$ is $y \leqslant -1$ or $y \geqslant 1$.

4 The graph of $y = \cot x$, $x \in \mathbb{R}$, has period 180° or π radians. It has vertical asymptotes at all the values of x for which $\tan x = 0$.

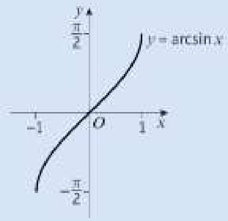


- The domain of $y = \cot x$ is $x \in \mathbb{R}$, $x \neq 0°, 180°, 360°, \ldots$ or any multiple of 180°.
- In radians the domain is $x \in \mathbb{R}$, $x \neq 0, \pi, 2\pi, \ldots$ or any multiple of π.
- The range of $y = \cot x$ is $y \in \mathbb{R}$.

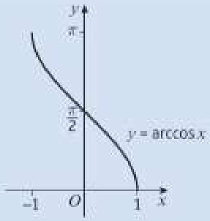
5 $\sec x = k$ and $\operatorname{cosec} x = k$ have no solutions for $-1 < k < 1$.

6 You can use the identity $\sin^2 x + \cos^2 x \equiv 1$ to prove the following identities:
- $1 + \tan^2 x \equiv \sec^2 x$
- $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$

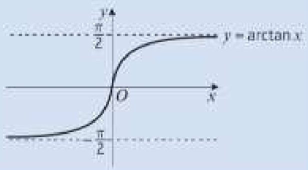
7 The inverse function of $\sin x$ is called **arcsin *x***.
- The domain of $y = \arcsin x$ is $-1 \leqslant x \leqslant 1$
- The range of $y = \arcsin x$ is $-\frac{\pi}{2} \leqslant \arcsin x \leqslant \frac{\pi}{2}$ or $-90° \leqslant \arcsin x \leqslant 90°$



8 The inverse function of $\cos x$ is called **arccos *x***.
- The domain of $y = \arccos x$ is $-1 \leqslant x \leqslant 1$
- The range of $y = \arccos x$ is $0 \leqslant \arccos x \leqslant \pi$ or $0° \leqslant \arccos x \leqslant 180°$



9 The inverse function of $\tan x$ is called **arctan *x***.
- The domain of $y = \arctan x$ is $x \in \mathbb{R}$
- The range of $y = \arctan x$ is $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$ or $-90° < \arctan x < 90°$



T.162 mixed Ex , P.47 BSG