KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 13
Applied Mathematics
P2 7 Trigonometry and modelling Booklet

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

Prior knowledge check

Prior knowledge check

1 Find the exact values of:

a $\sin 45°$ **b** $\cos \frac{\pi}{6}$ **c** $\tan \frac{\pi}{3}$ ← Section 5.4

2 Solve the following equations in the interval $0 \leqslant x < 360°$.

a $\sin(x + 50°) = -0.9$ **b** $\cos(2x - 30°) = \frac{1}{2}$

c $2\sin^2 x - \sin x - 3 = 0$ ← Year 1, Chapter 10

3 Prove the following:

a $\cos x + \sin x \tan x \equiv \sec x$ **b** $\cot x \sec x \sin x \equiv 1$

c $\dfrac{\cos^2 x + \sin^2 x}{1 + \cot^2 x} \equiv \sin^2 x$ ← Section 6.4

7.2) Using Angle Addition formulae

Addition Formulae allow us to deal with a **sum or difference of angles**.

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Do I need to memorise these?
They're all technically in the formula booklet, but you REALLY want to eventually memorise these (particularly the *sin* and *cos* ones).

How to memorise:

First notice that for all of these the first thing on the RHS is the same as the first thing on the LHS!

- For sin, the operator in the middle is the same as on the LHS.
- For cos, it's the opposite.
- For tan, it's the same in the numerator, opposite in the denominator.

- For sin, we mix sin and cos.
- For cos, we keep the cos's and sin's together.

Why is $\sin(A+B)$ not just $\sin(A) + \sin(B)$?

Because sin is a <u>function</u>, not a quantity that can be expanded out like this. It's a bit like how $(a+b)^2 \not\equiv a^2 + b^2$.
We can easily disprove it with a counterexample.

Notes

Worked Example

Using the trigonometric angle addition formulae find:

$\sin 75°$

$\tan 75°$

Worked Example

Using the trigonometric angle addition formulae find:

$\sin 15°$

$\tan 15°$

T.172 7B: Qs 1-2, P.49 7.3 Qs 1-2

Worked Example

Given that: $\sin A = \frac{8}{17}$ and $0° < A < 90°$, and $\cos B = -\frac{4}{5}$, B is obtuse, find the value of $\cos(A + B)$

Worked Example

Given that:

$\sin A = \frac{8}{17}$ and $0° < A < 90°$, and $\cos B = -\frac{4}{5}$, B is obtuse, find the value of $\tan(A - B)$

Worked Example

Given that: $\sin A = \frac{8}{17}$ and $0° < A < 90°$, and $\cos B = -\frac{4}{5}$, B is obtuse, find the value of $\sec(A - B)$

T.172 7B: Qs 3+, P.49 7.3 Qs 3+

Worked Example

Given that $2\cos(x-40)° = \sin(x-50)°$, show that $\tan x = 3\tan 50°$

T.172 7B: Qs 3+, P.49 7.3 Qs 3+

7.1) Addition formulae

Addition Formulae allow us to deal with a **sum or difference of angles**.

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Do I need to memorise these?
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How to memorise:

First notice that for all of these the first thing on the RHS is the same as the first thing on the LHS!

- For sin, the operator in the middle is the same as on the LHS.
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- For sin, we mix sin and cos.
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Why is $\sin(A+B)$ not just $\sin(A) + \sin(B)$?

Because sin is a function, not a quantity that can be expanded out like this. It's a bit like how $(a+b)^2 \not\equiv a^2 + b^2$.
We can easily disprove it with a counterexample.

Worked Example

Express the following as a single sine, cosine or tangent, and evaluate:

a) $\sin 30° \cos 60° + \cos 30° \sin 60°$

b) $\cos 20° \cos 25° - \sin 20° \sin 25°$

$c)$ $\dfrac{\tan\frac{\pi}{18} + \tan\frac{\pi}{9}}{1 - \tan\frac{\pi}{18}\tan\frac{\pi}{9}}$

T.169 7A: Qs 9

Worked Example

Write in the form $\sin(x \pm \theta)$ or $\cos(x \pm \theta)$ where $0 < \theta < \frac{\pi}{2}$:

$$\frac{1}{2}(\sqrt{3}\sin x + \cos x)$$

$$\frac{1}{2}(\sqrt{3}\cos x - \sin x)$$

Worked Example

Given that $\tan(x + \frac{\pi}{6}) = \frac{1}{2}$ evaluate $\tan x$

Given that $\tan(x - \frac{\pi}{3}) = \frac{1}{2}$ evaluate $\tan x$

T.169 7A: Qs 10+, P.49 7.1 Qs 3+

7.3) Double-angle formulae

Double-angle formula allow you to halve the angle within a trig function. **NOT IN FORMULAE BOOKLET**

$$\sin(2A) \equiv 2\sin A\cos A$$
$$\cos(2A) \equiv \cos^2 A - \sin^2 A$$
$$\equiv 2\cos^2 A - 1$$
$$\equiv 1 - 2\sin^2 A$$

This first form is relatively rare.

Fro Tip: The way I remember what way round these go is that the cos on the RHS is 'attracted' to the cos on the LHS, whereas the sin is pushed away.

$$\tan(2A) = \frac{2\tan A}{1-\tan^2 A}$$

These are all easily derivable by just setting $A = B$ in the compound angle formulae. e.g.

$$\sin(2A) = \sin(A+A)$$
$$= \sin A\cos A + \cos A\sin A$$
$$= 2\sin A\cos A$$

TASK: derive other two

Notes

can learn short hand as:

$$c2x \ = \ c^2 - s^2 s2x = 2sc$$

You then get all the alternatives for $c2x$ by swapping:

$$c^2 \ for \ 1 - s^2$$

or

$$s^2 \ for \ 1 - c^2$$

**Later you will need to be able to rearrange the cos2x identities to make $c^2 \ or \ s^2$ the subjects too*

Worked Example

Use the double-angle formulae to write as a single trigonometric ratio:

a) $\cos^2 50° - \sin^2 50°$ b) $2\cos^2\frac{2\pi}{9} - 1$ c) $1 - 2\sin^2 30°$

Worked Example

Use the double-angle formulae to write as a single trigonometric ratio:

a) $\cos^2 2x - \sin^2 2x$ b) $4\cos^2 3x - 2$ c) $3 - 6\sin^2 4x$

Worked Example

Use the double-angle formulae to write as a single trigonometric ratio:

a) $2\sin 45° \cos 45°$ b) $4\sin\frac{\pi}{12}\cos\frac{\pi}{12}$ c) $7\sin 5x \cos 5x$

Worked Example

Use the double-angle formulae to write as a single trigonometric ratio:

a) $\frac{2\tan 30°}{1-\tan^2 30°}$ $\quad$ $b)$ $\frac{2\tan\frac{\pi}{12}}{1-\tan^2\frac{\pi}{12}}$ $\quad$ c) $\frac{4\tan 6x}{1-\mathrm{ta}\ {}^2\,6x}$

Worked Example

Use the double-angle formulae to write as a single trigonometric ratio:

a) $\frac{8 \sin 22.5°}{\sec 22.5°}$ b) $\frac{6 \cos\frac{\pi}{4}}{\text{cosec}\frac{\pi}{4}}$

T.175 7C: Qs 4-8, P.50 7.3 Qs 1-5

Worked Example

Given that $x = 2\sin\theta$ and $y = 4 - 3cos2\theta$, eliminate θ and express y in terms of x.

Worked Example

Given that $\cos x = \frac{5}{8}$ and x is acute, find the exact value of

(a) $\sin 2x$ (b) $\tan 2x$

Worked Example

Using the double-angle formulae, evaluate:

a) $\left(\sin\frac{\pi}{3}+\cos\frac{\pi}{3}\right)^2$ b) $\left(\sin\frac{\pi}{4}-\cos\frac{\pi}{4}\right)^2$

Worked Example

Given that $0 < \theta < \pi$, find the value of $\tan\frac{\theta}{2}$ when $\tan\theta = -\frac{3}{4}$

T.175 7C: Qs 9+, P.50 7.3 Qs 6+

10. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\cos 3A \equiv 4\cos^3 A - 3\cos A$$

(4)

(b) Hence solve, for $-90° \leqslant x \leqslant 180°$, the equation

$$1 - \cos 3x = \sin^2 x$$

(4)

Your Turn

10. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(*a*) Show that

$$\sin 3A \equiv 3\sin A - 4\sin^3 A$$

(4)

(*b*) Hence solve, for $0° \leq x \leq 180°$, the equation

$$1 - \sin 3x = \cos^2 x$$

(4)

(Total for Question 10 is 8 marks)

10. **In this question you should show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Given that $1 + \cos 2\theta + \sin 2\theta \neq 0$ prove that

$$\frac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta} \equiv \tan\theta$$

(4)

(b) Hence solve, for $0 < x < 180°$

$$\frac{1 - \cos 4x + \sin 4x}{1 + \cos 4x + \sin 4x} = 3\sin 2x$$

giving your answers to one decimal place where appropriate.

(4)

Your Turn

10. **In this question you should show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(*a*) Given that $1+\cos 2\theta+\sin 2\theta \neq 0$ prove that

$$\frac{\cos 2\theta - 1 - \sin 2\theta}{1+\cos 2\theta + \sin 2\theta} = -\tan\theta$$

(4)

(*b*) Hence solve, for $0 < x < 180°$

$$\frac{\cos 6x - 1 - \sin 6x}{1+\cos 6x + \sin 6x} = 4\sin 3x$$

giving your answers to one decimal place where appropriate.

(4)

(Total for Question 10 is 8 marks)

7.4) Solving trigonometric equations

Combine your knowledge from *Pure 1 trigonometry:*

- Finding multiple angles rules
- Using the domain given in the question

With the trig. Identities:

Trigonometric identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad (A \pm B \neq (k + \tfrac{1}{2})\pi)$$

and

$$\sin(2A) \equiv 2 \sin A \cos A$$

$$\cos(2A) \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$

$$\tan(2A) = \frac{2 \tan A}{1 - \tan^2 A}$$

Notes

Worked Example

Solve in the interval $0 \leq x \leq 360°$: $8\sin(\theta + 60°) = 4\sqrt{2}\cos\theta$

Worked Example

Solve in the interval $0 \leq x \leq 360°$: $8\cos(\theta - 60°) = 4\sqrt{2}\sin\theta$

T.180 7D: Qs 1-4, P.51 7.4 Qs 1-5

Worked Example

Solve in the interval $0 \leq x \leq 360°$: $3\cos 2x + \cos x + 2 = 0$

Worked Example

Solve in the interval $0 \leq x \leq 360°$: $3\cos 2x - \sin x - 2 = 0$

Worked Example

Solve in the interval $0 \leq x \leq 360°$: $4\sin 2x - 5\cos x = 0$

Worked Example

Solve in the interval $0 \leq y \leq 2\pi$: $3\tan 2y \tan y = 2$

T.180 7D: Qs 5-10, P.51 7.4 Qs 6

Worked Example

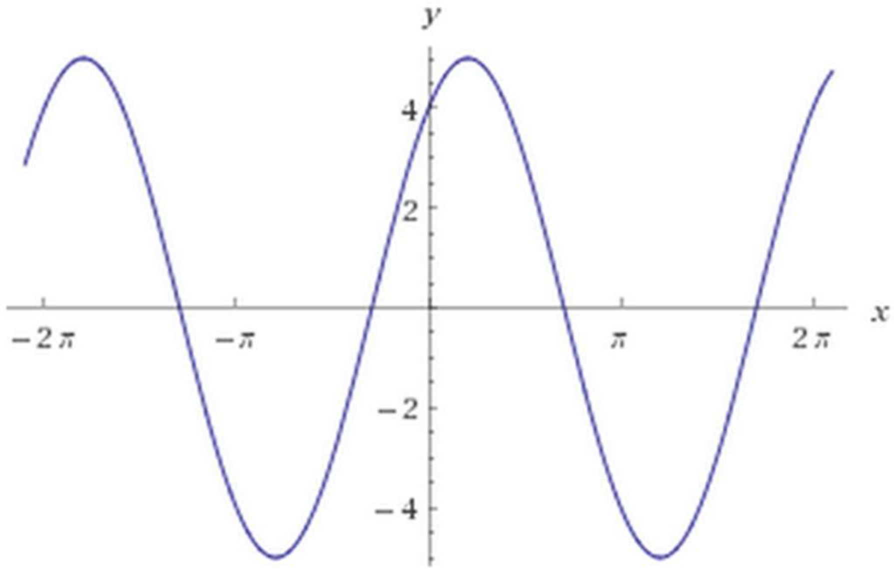
a) Show that $\cos(3A) = 4\cos^3 A - 3\cos A$.

b) Hence or otherwise, solve, for $0 < \theta < 2\pi$, the equation $12\cos\theta - 16\cos^3\theta - 2\sqrt{3} = 0$

T.180 7D: Qs 11+, P.51 7.4 Qs 7+

7.5) Simplifying $a\cos x \pm b\sin x$

Here's a sketch of $y = 3\sin x + 4\cos x$.



It's a sin graph that seems to be translated on the x-axis and stretched on the y axis. This suggests we can represent it as $y = R\sin(x + \alpha)$, where α is the horizontal translation and R the stretch on the y-axis.

Notes

Put $3\sin x + 4\cos x$ in the form $R\sin(x+\alpha)$ giving α in degrees to 1dp.

STEP 1: Expanding:

$$R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$$

STEP 2: Comparing coefficients:

$$R\cos\alpha = 3 \qquad R\sin\alpha = 4$$

STEP 3: Using the fact that $R^2\sin^2\alpha + R^2\cos^2\alpha = R^2$:

$$R = \sqrt{3^2+4^2} = 5$$

If $R\cos\alpha = 3$ and $R\sin\alpha = 4$ then $R^2\cos^2\alpha = 3^2$ and $R^2\sin^2\alpha = 4^2$.

$R^2\sin^2\alpha + R^2\cos^2\alpha = 3^2+4^2$

$R^2(\sin^2\alpha + \cos^2\alpha) = 3^2+4^2$

$R^2 = 3^2+4^2$

$R = \sqrt{3^2+4^2}$

(You can write just the last line in exams)

STEP 4: Using the fact that $\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha$:

$$\tan\alpha = \frac{4}{3}$$

$$\alpha = 53.1°$$

STEP 5: Put values back into original expression.

$$3\sin x + 4\cos x \equiv 5\sin(x+53.1°)$$

Worked Example

591a: Write $a\cos x + b\sin x$ in the form $R\cos(x+\alpha)$

Write $9\sin\theta + 3\cos\theta$ in the form $R\sin(\theta+\alpha)$ leaving R as an exact value and α in degrees correct to 1 decimal place.

You may use these formulae:

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$
$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

SLOP the routine

1 Express each of the following in the form $R\cos(x-\alpha)°$, where $R>0$ and $0<\alpha<90$. Give the values of R and α correct to 1 decimal place where appropriate.

a $\cos x° + \sin x°$ **b** $3\cos x° + 4\sin x°$

c $2\sin x° + \cos x°$ **d** $\cos x° + \sqrt{3}\sin x°$

2 Express each of the following in the given form, where $R>0$ and $0<\alpha<90$. Give the exact value of R and the value of α correct to 1 decimal place.

a $5\cos x° - 12\sin x°$, $R\cos(x+\alpha)°$ **b** $4\sin x° + 2\cos x°$, $R\sin(x+\alpha)°$

c $\sin x° - 7\cos x°$, $R\sin(x-\alpha)°$ **d** $8\cos 2x° - 15\sin 2x°$, $R\cos(2x+\alpha)°$

3 Express each of the following in the given form, where $R>0$ and $0<\alpha<\frac{\pi}{2}$. Give the exact value of R and the value of α correct to 2 decimal places where appropriate.

a $3\sin x - 2\cos x$, $R\sin(x-\alpha)$ **b** $3\cos x + \sqrt{3}\sin x$, $R\cos(x-\alpha)$

c $8\sin 3x + 6\cos 3x$, $R\sin(3x+\alpha)$ **d** $\cos x + \frac{1}{2}\sin x$, $R\cos(x-\alpha)$

4 Find the maximum value that each expression can take and the smallest positive value of x, in degrees, for which this occurs.

a $24\sin x - 7\cos x$ **b** $4\cos 2x + 4\sin 2x$

c $3\cos x - 5\sin x$ **d** $5\sin 3x + \cos 3x$

5 **a** Express $3\sin x° - 3\cos x°$ in the form $R\sin(x-\alpha)°$, where $R>0$ and $0<\alpha<90$.

b Hence, describe two transformations that would map the graph of $y=\sin x°$ onto the graph of $y = 3\sin x° - 3\cos x°$.

6 By first expressing each curve in an appropriate form, sketch each of the following for x in the interval $0 \le x \le 360°$, showing the coordinates of any turning points.

a $y = 12\cos x + 5\sin x$ **b** $y = \sin x - 2\cos x$

c $y = 2\sqrt{3}\cos x - 6\sin x$ **d** $y = 9\sin x + 4\cos x$

7 **a** Express $\sqrt{3}\cos x - \sin x$ in the form $R\cos(x+\alpha)$, where $R>0$ and $0<\alpha<\frac{\pi}{2}$.

b Solve the equation $\sqrt{3}\cos x - \sin x = 1$ for x in the interval $0 \le x \le 2\pi$, giving your answers in terms of π.

8 Solve each equation for x in the interval $0 \le x \le 2\pi$, giving your answers to 2 decimal places.

a $6\sin x + 8\cos x = 5$ **b** $2\cos x - 2\sin x = 1$

c $7\sin x - 24\cos x - 10 = 0$ **d** $3\cos x + \sin x + 1 = 0$

e $\cos 2x + 4\sin 2x = 3$ **f** $5\sin x - 8\cos x + 7 = 0$

9 Solve each equation for x in the interval $-180° \le x \le 180°$, giving your answers to 1 decimal place where appropriate.

a $\sin x + \cos x = 1$ **b** $4\cos x - \sin x + 2 = 0$

c $\cos\frac{x}{2} + 5\sin\frac{x}{2} - 4 = 0$ **d** $6\sin x = 5 - 3\cos x$

T.184 7E: Qs 1-4, P.52 7.5 Qs 1-3

Worked Example

Solve in the interval $0 < \theta < 360°$:

$$5\cos\theta + 2\sin\theta = 3$$

Worked Example

Solve in the interval $0 \leq \theta < 180°$:

$$5 \sin 3\theta - 12 \cos 3\theta = 1$$

Worked Example

Solve in the interval $0 \leq \theta < 360°$:

$$\cot \theta + 4 = cosec\ \theta$$

T.184 7E: Qs 5-8, P.52 7.5 Qs 4-5

Worked Example

Find the maximum value and the smallest positive value of θ at which the maximum occurs for:

$$4\cos\theta + 3\sin\theta$$

Worked Example

Find the maximum value and the smallest positive value of θ at which the maximum occurs for:

$$\frac{5}{7 + 4\cos\theta - 3\sin\theta}$$

Worked Example

Find the minimum value and the smallest positive value of θ at which the minimum occurs for:

$$\frac{5}{7 + 4\cos\theta - 3\sin\theta}$$

T.184 7E: Qs 9+, P.52 7.5 Qs 6+

8. (a) Express $2\cos\theta + 8\sin\theta$ in the form $R\cos(\theta - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α in radians to 3 decimal places.

(3)

The first three terms of an arithmetic sequence are

$$\cos x \qquad \cos x + \sin x \qquad \cos x + 2\sin x \qquad x \neq n\pi$$

Given that S_9 represents the sum of the first 9 terms of this sequence as x varies,

(b) (i) find the exact maximum value of S_9

(ii) deduce the smallest positive value of x at which this maximum value of S_9 occurs.

(3)

Your Turn

8. (a) Express $\cos\theta + 8\sin\theta$ in the form $R\cos(\theta - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$

Give the exact value of R and give the value of α in radians to 3 decimal places.

(3)

The first three terms of an arithmetic sequence are

$$\cos x \qquad \cos x + 2\sin x \qquad \cos x + 4\sin x \qquad x \neq n\pi$$

Given that S_9 represents the sum of the first 9 terms of this sequence as x varies,

(b) (i) find the exact maximum value of S_9

(ii) deduce the smallest positive value of x at which this maximum value of S_9 occurs.

(3)

(Total for Question 8 is 6 marks)

7.6) Proving trigonometric identities

REMEMBER layout proof as:

$$
\begin{aligned}
LHS &\equiv \ldots \\
&\equiv \ldots \\
&\equiv \ldots \\
&\equiv RHS
\end{aligned}
$$

Double Angle

$\sin 2x = 2\sin x\cos x$
$\cos 2x = \cos^2 x - \sin^2 x$ $\cos 2x = 2\cos^2 x - 1$ $\cos 2x = 1 - 2\sin^2 x$
$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

Notes

Worked Example

Prove that:

$$\cot 2\theta \equiv \frac{\cot\theta - \tan\theta}{2}$$

Double Angle

$\sin 2x = 2\sin x\cos x$
$\cos 2x = \cos^2 x - \sin^2 x$ $\cos 2x = 2\cos^2 x - 1$ $\cos 2x = 1 - 2\sin^2 x$
$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$

Worked Example

Prove that:

$$\frac{-\sin 2\theta}{\cos 2\theta - 1} \equiv \cot\theta$$

Worked Example

Prove that:

$$\cot 2x - cosec\ 2x \equiv -\tan x$$

Worked Example

Prove, starting with the left-hand side:

$$\tan 2x + sec\ 2x \equiv \frac{\cos x + \sin x}{\cos x - \sin x}$$

T.187 7F: Qs 1-5, P.53 7.6 Qs 1-6

Worked Example

Show that:

$$\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$$

Worked Example

By writing $\cos x = \cos\left(2 \times \frac{x}{2}\right)$, prove the identity

$$\frac{1 + \cos x}{1 - \cos x} \equiv \cot^2\left(\frac{x}{2}\right)$$

T.187 7F: Qs 6+, P.53 7.6 Qs 7

12. (a) Prove

$$\frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta} \equiv 2\cot 2\theta \qquad \theta \neq (90n)^\circ,\ n \in \mathbb{Z}$$

(4)

(b) Hence solve, for $90^\circ < \theta < 180^\circ$, the equation

$$\frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta} = 4$$

giving any solutions to one decimal place.

(3)

Your Turn

12. (*a*) Prove that

$$\frac{\cos 3\theta}{3\sin\theta} + \frac{\sin 3\theta}{3\cos\theta} = \frac{2}{3}\cot 2\theta, \qquad \theta \neq (90n)°, \quad n \in \mathbb{Z}.$$

(4)

(*b*) Hence solve, for $0° < \theta < 180°$, the equation

$$\frac{\cos 3\theta}{3\sin\theta} + \frac{\sin 3\theta}{3\cos\theta} = 1$$

giving any solutions to one decimal place.

(3)

(Total for Question 12 is 7 marks)

7.7) Modelling with trigonometric functions

Notes

Worked Example

The cabin pressure, P (psi) on an aeroplane at cruising altitude can be modelled by the equation

$$P = 14.5 - 0.2\sin(t - 3)$$

where t is the time in hours since cruising altitude was first reached, and angles are in radians. Find:

a) The maximum and minimum cabin pressure
b) The time after reaching cruising altitude that the cabin first reaches a maximum pressure
c) The cabin pressure after 3 hours at cruising altitude
d) All the times within the first 10 hours of cruising that the cabin pressure would be exactly 14.42 psi

Worked Example

a) Express $7\cos\theta - 5\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. State the exact value of R and give α to four decimal places.

b) State the maximum value of $7\cos\theta - 5\sin\theta$ and the value of θ, for $0 < \theta < 2\pi$ at which this maximum occurs.

The height H above ground of a passenger on a Ferris wheel is modelled by the equation

$$H = 12 - 7\cos(\frac{\pi t}{4}) + 5\sin(\frac{\pi t}{4})$$

where H is measured in metres, and t is the time in minutes after the wheel starts turning.

c) Calculate the maximum value of H predicted by this model, and the value of t when this maximum first occurs

d) Determine the time for the Ferris wheel to complete five revolutions

Worked Example

a) Express $1.5 \sin\theta - 2\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. State the exact value of R and give α to four decimal places.

b) State the maximum value of $1.5 \sin\theta - 2\cos\theta$ and the value of θ, for $0 < \theta < \pi$ at which this maximum occurs.

The height H of sea water on a particular day can be modelled by the equation

$$H = 8 + 1.5\sin\left(\frac{2\pi t}{25}\right) - 2\cos\left(\frac{2\pi t}{25}\right), 0 \leq t < 12$$

where H is measured in metres, and t is the number of hours after midnight.

c) Calculate the maximum value of H predicted by this model, and the value of t when this maximum occurs

d) Calculate, to the nearest minute, the times when the height of sea water is predicted, by this model, to be 7 metres.

T.190 7G: Qs all, P.54 7.7 Qs all

15. (a) Express $2\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and the value of α in radians to 3 decimal places.

(3)

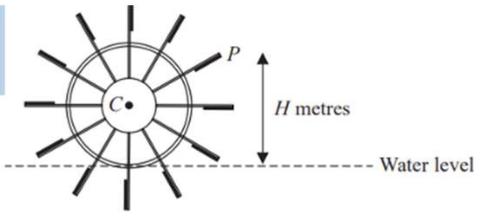


Figure 6

Figure 6 shows the cross-section of a water wheel.

The wheel is free to rotate about a fixed axis through the point C.

The point P is at the end of one of the paddles of the wheel, as shown in Figure 6.

The water level is assumed to be horizontal and of constant height.

The vertical height, H metres, of P above the water level is modelled by the equation

$$H = 3 + 4\cos(0.5t) - 2\sin(0.5t)$$

where t is the time in seconds after the wheel starts rotating.

Using the model, find

(b) (i) the maximum height of P above the water level,

(ii) the value of t when this maximum height first occurs, giving your answer to one decimal place.

(3)

In a single revolution of the wheel, P is below the water level for a total of T seconds.

According to the model,

(c) find the value of T giving your answer to 3 significant figures.

(*Solutions based entirely on calculator technology are not acceptable.*)

(4)

In reality, the water level may not be of constant height.

(d) Explain how the equation of the model should be refined to take this into account.

(1)

Your Turn

15. (*a*) Express $3\cos\theta - 2\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and the value of α in radians to 3 decimal places.

(3)

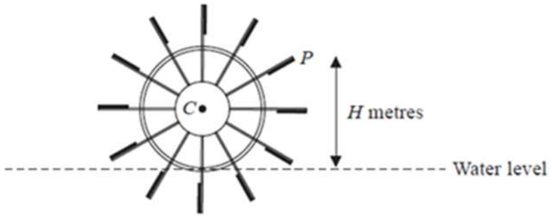


Figure 6

Figure 6 shows the cross-section of a water wheel.

The wheel is free to rotate about a fixed axis through the point C.

The point P is at the end of one of the paddles of the wheel, as shown in Figure 6.

The water level is assumed to be horizontal and of constant height.

The vertical height, H metres, of P above the water level is modelled by the equation

$$H = 5 + 9\cos(0.2t) - 6\sin(0.2t)$$

where t is the time in seconds after the wheel starts rotating.

Using the model, find

(*b*) (i) the maximum height of P above the water level,

(ii) the value of t when this maximum height first occurs, giving your answer to one decimal place.

(3)

In a single revolution of the wheel, P is below the water level for a total of T seconds.

According to the model,

(*c*) find the value of T giving your answer to 3 significant figures.

(*Solutions based entirely on calculator technology are not acceptable*.)

(4)

In reality, the water level may not be of constant height.

(*d*) Explain how the equation of the model should be refined to take this into account.

(1)

(Total for Question 15 is 11 marks)

9.

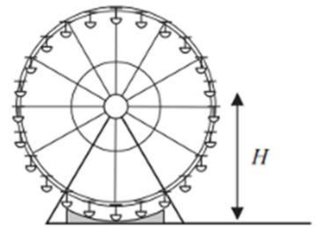


Figure 4

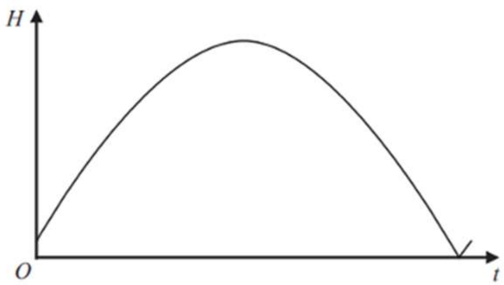


Figure 5

Figure 4 shows a sketch of a Ferris wheel.

The height above the ground, H m, of a passenger on the Ferris wheel, t seconds after the wheel starts turning, is modelled by the equation

$$H = |A\sin(bt + \alpha)°|$$

where A, b and α are constants.

Figure 5 shows a sketch of the graph of H against t, for one revolution of the wheel.

Given that

- the maximum height of the passenger above the ground is 50 m
- the passenger is 1 m above the ground when the wheel starts turning
- the wheel takes 720 seconds to complete one revolution

(a) find a complete equation for the model, giving the exact value of A, the exact value of b and the value of α to 3 significant figures.

(4)

(b) Explain why an equation of the form

$$H = |A\sin(bt + \alpha)°| + d$$

where d is a positive constant, would be a more appropriate model.

(1)

9.

Figure 4

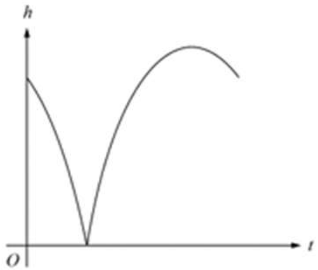


Figure 5

Figure 4 shows a sketch of child's toy.

The toy consists of a specially weighted wheel that moves along the floor. The diameter of the wheel is 40 cm.

An arrow is drawn on the wheel. At the instant when the wheel starts to move, the point of the arrow is 35 cm above the floor.

The wheel takes 2 seconds to complete one revolution.

The height above the ground, h cm, of the point of the arrow, t seconds after the wheel starts to move, is modelled by the equation

$$h = |A\cos(bt + \alpha)°|$$

where A, b and α are constants.

Figure 5 shows a sketch of the graph of h against t, for one revolution of the wheel.

(a) find a complete equation for the model, giving the exact value of A, the exact value of b and the value of α to 3 significant figures.

(4)

On another occasion the wheel is positioned so that the arrow is pointing vertically downwards at the instant when the wheel starts to move.

(b) Write down an equation for h of the form

$$h = |A\cos(bt + \alpha)°|$$

stating the values of each of the constants A, b and α.

(1)

(Total for Question 9 is 5 marks)

Extract from Formulae book

Trigonometric identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad (A \pm B \neq (k + \tfrac{1}{2})\pi)$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Small angle approximations

$$\sin\theta \approx \theta$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$

$$\tan\theta \approx \theta$$

where θ is measured in radians

13. (a) Show that

$$\operatorname{cosec} 2x + \cot 2x \equiv \cot x, \qquad x \neq 90n°,\ n \in \mathbb{Z}$$

(5)

(b) Hence, or otherwise, solve, for $0 \leqslant \theta < 180°$,

$$\operatorname{cosec}(4\theta + 10°) + \cot(4\theta + 10°) = \sqrt{3}$$

You must show your working.
(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(5)

Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

Question	Scheme	Marks	AOs
13(a)	$\operatorname{cosec}2x + \cot 2x \equiv \cot x,\ x \neq 90n°,\ n \in \mathbb{Z}$		
	$\operatorname{cosec}2x + \cot 2x = \frac{1}{\sin 2x} + \frac{\cos 2x}{\sin 2x}$	M1	1.2
	$= \frac{1+\cos 2x}{\sin 2x}$	M1	1.1b
	$= \frac{1+2\cos^2 x - 1}{2\sin x\cos x} = \frac{2\cos^2 x}{2\sin x \cos x}$	M1 A1	2.1 1.1b
	$= \frac{\cos x}{\sin x} = \cot x$ *	A1*	2.1
		(5)	
(b)	$\operatorname{cosec}(4\theta+10°) + \cot(4\theta+10°) = \sqrt{3};\ 0 \leqslant \theta < 180°$		
	$\cot(2\theta \pm ...°) = \sqrt{3}$	M1	2.2a
	$2\theta \pm ... = 30° \Rightarrow \theta = 12.5°$	M1 A1	1.1b 1.1b
	$2\theta + 5° = 180° + \text{PV}° \Rightarrow \theta = ...°$	M1	2.1
	$\theta = 102.5°$	A1	1.1b
		(5)	
		(10 marks)	

Summary of Key Points

Summary of key points

1 The **addition** (or compound-angle) formulae are:

- $\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$ $\quad$ $\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$
- $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$ $\quad$ $\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$
- $\tan(A + B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ $\quad$ $\tan(A - B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

2 The **double-angle** formulae are:

- $\sin 2A \equiv 2 \sin A \cos A$
- $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
- $\tan 2A \equiv \dfrac{2\tan A}{1 - \tan^2 A}$

3 For positive values of a and b,

- $a \sin x \pm b \cos x$ can be expressed in the form $R \sin(x \pm \alpha)$
- $a \cos x \pm b \sin x$ can be expressed in the form $R \cos(x \mp \alpha)$

with $R > 0$ and $0 < \alpha < 90°$ $\left(\text{or } \frac{\pi}{2}\right)$

where $R \cos \alpha = a$ and $R \sin \alpha = b$ and $R = \sqrt{a^2 + b^2}$.

T.192 mixed ex, P.57 BSG