KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 12
Applied Mathematics
P2 12 Vectors Booklet

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

Prior knowledge check

Prior knowledge check

1 Given that $\mathbf{p} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{q} = -\mathbf{i} + 2\mathbf{j}$, calculate:

a $2\mathbf{p} + \mathbf{q}$ **b** $-3\mathbf{p} + 4\mathbf{q}$

← Year 1, Section 11.2

2 Given that $\mathbf{a} = 5\mathbf{i} - 3\mathbf{j}$, work out:

a the magnitude of $\mathbf{a}$

b the unit vector that is parallel to $\mathbf{a}$.

← Year 1, Section 11.3

3 M is the midpoint of the line segment AB.

Given that $\overrightarrow{AB} = 4\mathbf{i} + \mathbf{j}$,

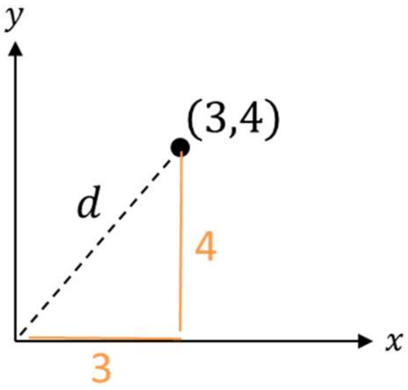
a find $\overrightarrow{BM}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

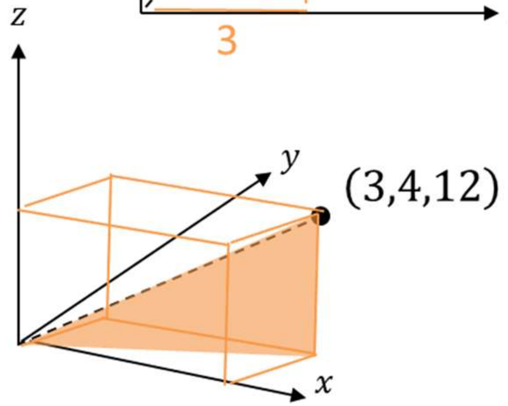
The point P lies on AB such that $AP : PB = 3 : 1$.

b Find $\overrightarrow{AP}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

← Year 1, Section 11.5

12.1) 3D coordinates





In 2D, how did we find the distance from a point to the origin?

Using Pythagoras:

$$\boldsymbol{d = \sqrt{3^2 + 4^2} = 5}$$

How about in 3D then?

You may be familiar with this method from GCSE.

Using Pythagoras on the base of the cuboid:

$$\boldsymbol{\sqrt{3^2 + 4^2} = 5}$$

Then using the highlighted triangle:

$$\boldsymbol{\sqrt{5^2 + 12^2} = 13}$$

We could have similarly done this is one go using:

$$\boldsymbol{\sqrt{3^2 + 4^2 + 12^2} = 13}$$

From Year 1 you will be familiar with the magnitude $|\boldsymbol{a}|$ of a vector $\boldsymbol{a}$ being its length.
We can see from above that this nicely extends to 3D:

The magnitude of a vector $\boldsymbol{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$:

$$|\boldsymbol{a}| = \sqrt{x^2 + y^2 + z^2}$$

And the distance of (x, y, z) from the origin is $\sqrt{x^2 + y^2 + z^2}$

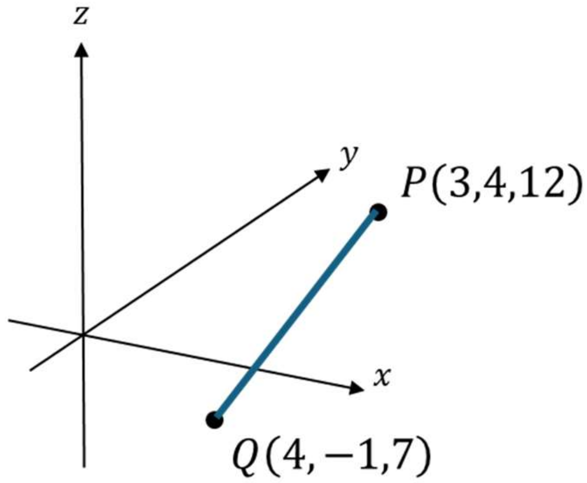
Notes

Worked Example

Find the distance from the origin to the point with coordinates $(6, 8, 24)$

Find the distance from the origin to the point with coordinates $(-6, 0, -2)$

Distance between two 3D points



How do we find the distance between P and Q?

It's just the magnitude/length of the vector between them.

i.e.

$$|\overrightarrow{\boldsymbol{PQ}}| = \left|\begin{pmatrix} \mathbf{1} \\ \mathbf{-5} \\ \mathbf{-5} \end{pmatrix}\right|$$

$$= \sqrt{\mathbf{1^2 + (-5)^2 + (-5)^2}} = \sqrt{\mathbf{51}}$$

✏ The distance between two points is:

$$d = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

dx means "change in

Quickfire Questions:

Distance of $(4,0,-2)$ from the origin:

$$\sqrt{\mathbf{4^2 + 0^2 + (-2)^2}} = \sqrt{\mathbf{20}}$$

$$\left|\begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix}\right| = \sqrt{\mathbf{5^2 + 4^2 + (-1)^2}} = \sqrt{\mathbf{42}}$$

Distance between $(0,4,3)$ and $(5,2,3)$.

$$\boldsymbol{d} = \sqrt{\mathbf{5^2 + (-2)^2 + 0^2}} = \sqrt{\mathbf{29}}$$

Distance between $(1,1,1)$ and $(2,1,0)$.

$$\boldsymbol{d} = \sqrt{\mathbf{1^2 + 0^2 + 1^2}} = \sqrt{\mathbf{2}}$$

Distance between $(-5,2,0)$ and $(-2,-3,-3)$.

$$\boldsymbol{d} = \sqrt{\mathbf{3^2 + 5^2 + 3^2}} = \sqrt{\mathbf{43}}$$

Tip: Because we're squaring, it doesn't matter whether the change is negative or

Worked Example

The coordinates of A and B are $(3,5,-2)$ and $(3,k,-1)$ respectively. Given that the distance from A to B is $\sqrt{2}$ units, find the possible values of k.

12.2) Vectors in 3D

In 2D you were previously introduced to $\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as unit vectors in each of the x and y directions.

It meant for example that $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$ could be written as $8\boldsymbol{i} - 2\boldsymbol{j}$ since $8\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$

Unsurprisingly, in 3D:

$$\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Worked Example

Consider the points $A(-1, -5, 2)$ and $B(-7, 3, 0)$.

a) Find the position vectors of A and B in ijk notation

b) Find the vector $\overrightarrow{AB}$ as a column vector.

Worked Example

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are given by:

$$\boldsymbol{a} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$$

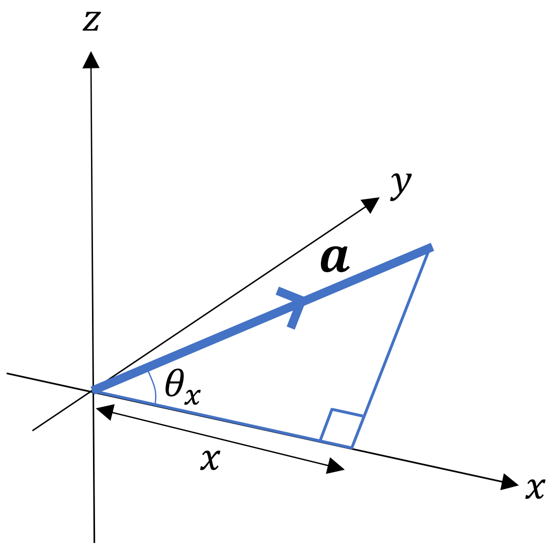
a) Find:

i) $\boldsymbol{a} + 3\boldsymbol{b}$

ii) $4\boldsymbol{a} - 5\boldsymbol{b}$

b) State whether these vectors are parallel to $-4\boldsymbol{i} + 16\boldsymbol{j}$

Angles between vectors and an axis



How could you work out the angle between a vector and the x-axis?

Just form a right-angle triangle!

The angle between $\boldsymbol{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and the x-axis is:

$$\cos \theta_x = \frac{x}{|\boldsymbol{a}|}$$

and similarly for the y and z axes.

Worked Example

Find the angles that the vector

$$\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$$

makes with each of the positive coordinate axes. Give your answers to 1 decimal place.

Worked Example

The points A and B have position vectors
$\boldsymbol{i} + 5\boldsymbol{j} + 3\boldsymbol{k}$ and $-2\boldsymbol{i} + 4\boldsymbol{j} + 8\boldsymbol{k}$ relative to a fixed origin, O.
Show that ΔOAB is isosceles.

Worked Example

$$\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k} \text{ and } \boldsymbol{b} = \boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$$

By considering the angles that $\boldsymbol{a}$ and $\boldsymbol{b}$ make with the x-axis, determine the area of OAB where $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$.

Worked Example

A triangle PQR is such that
$\overrightarrow{PQ} = -2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ and $\overrightarrow{QR} = 4\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}$
Find $\angle PQR$ to 1 decimal place

12.3) Solving geometric problems

Worked Example

A, B, C and D are the points $(3, -4, -9)$, $(1, -7, -3)$, $(1, 0, -15)$ and $(7, 9, -33)$ respectively.

a) Find $\overrightarrow{AB}$ and $\overrightarrow{DC}$, giving your answers in the form $p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k}$.

b) Show that the lines AB and DC are parallel and that $\overrightarrow{DC} = 3\overrightarrow{AB}$.

c) Hence describe the quadrilateral $ABCD$.

Worked Example

P, Q and R are the points $(9,3,-4)$, $(-5,5,5)$ and $(0,2,-8)$ respectively.
Find the coordinates of the point S so that $PQRS$ forms a parallelogram.

Comparing Coefficients

There are many contexts in maths where we can 'compare coefficients', e.g.

$3x^2 + 5x \equiv A(x^2 + 1) + Bx + C$
Comparing x^2 terms: $3 = A$

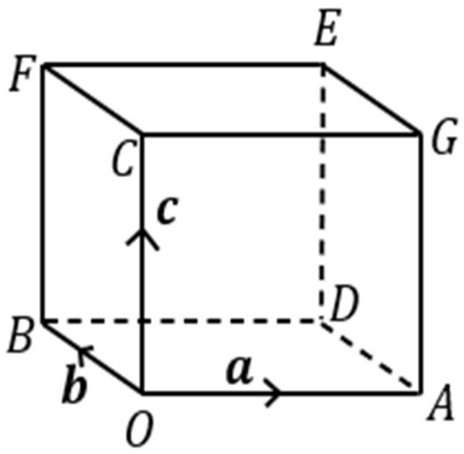
We can do the same with vectors:

Given that
$3\boldsymbol{i} + (p + 2)\boldsymbol{j} + 120\boldsymbol{k} = p\boldsymbol{i} - q\boldsymbol{j} + 4pqr\boldsymbol{k}$,
find the values of p, q and r.

Comparing i: $3 = p$
Comparing j: $p + 2 = -q \quad \therefore q = -5$
Comparing k: $120 = 4pqr \quad \therefore r = -2$

Worked Example

The diagram shows a cuboid whose vertices are O, A, B, C, D, E, F and G.
Vectors a, b and c are the position vectors of the vertices A, B and C respectively.
Prove that the diagonals OE and AF bisect each other.



3. Relative to a fixed origin O

- point A has position vector $2\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$
- point B has position vector $3\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$
- point C has position vector $2\mathbf{i} - 16\mathbf{j} + 4\mathbf{k}$

(a) Find $\overrightarrow{AB}$

(2)

(b) Show that quadrilateral $OABC$ is a trapezium, giving reasons for your answer.

(2)

Your Turn

3. Relative to a fixed origin O

- point P has position vector $3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$
- point Q has position vector $5\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$
- point R has position vector $4\mathbf{i} - 10\mathbf{j} + 6\mathbf{k}$

(*a*) Find $\overrightarrow{PQ}$

(2)

(*b*) Show that quadrilateral $OPQR$ is a trapezium, giving reasons for your answer.

(2)

(Total for Question 3 is 4 marks)

6.

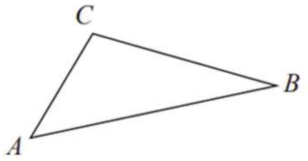


Figure 1

Figure 1 shows a sketch of triangle ABC.

Given that

- $\overrightarrow{AB} = -3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$
- $\overrightarrow{BC} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}$

(a) find $\overrightarrow{AC}$

(2)

(b) show that $\cos ABC = \dfrac{9}{10}$

(3)

Your Turn

6.

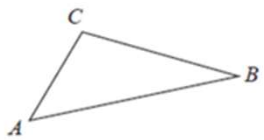


Figure 1

Figure 1 shows a sketch of triangle ABC.

Given that

- $\overrightarrow{AB} = -2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$
- $\overrightarrow{BC} = 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

(*a*) find $\overrightarrow{AC}$

(2)

(*b*) show that $\cos ABC = 0$

(3)

(Total for Question 6 is 5 marks)

9.

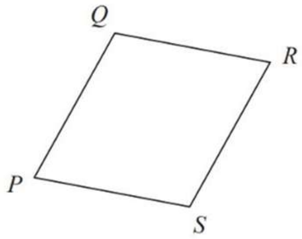


Figure 3

Figure 3 shows a sketch of a parallelogram $PQRS$.

Given that

- $\overrightarrow{PQ} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$
- $\overrightarrow{QR} = 5\mathbf{i} - 2\mathbf{k}$

(a) show that parallelogram $PQRS$ is a rhombus.

(2)

(b) Find the exact area of the rhombus $PQRS$.

(4)

Your Turn

9.

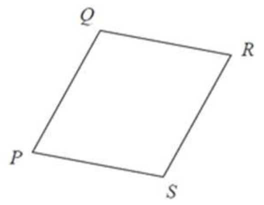


Figure 3

Figure 3 shows a sketch of a parallelogram *PQRS*.

Given that

- $\overrightarrow{PQ} = 5\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$
- $\overrightarrow{QR} = \mathbf{j} - 7\mathbf{k}$

(*a*) show that parallelogram *PQRS* is a rhombus.

(2)

(*b*) Find the exact area of the rhombus *PQRS*.

(4)

(Total for Question 9 is 6 marks)

13. Relative to a fixed origin O

- the point A has position vector $4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$
- the point B has position vector $4\mathbf{j} + 6\mathbf{k}$
- the point C has position vector $-16\mathbf{i} + p\mathbf{j} + 10\mathbf{k}$

where p is a constant.

Given that A, B and C lie on a straight line,

(a) find the value of p.

The line segment OB is extended to a point D so that $\overrightarrow{CD}$ is parallel to $\overrightarrow{OA}$

(b) Find $\left|\overrightarrow{OD}\right|$, writing your answer as a fully simplified surd.

Your Turn

13. Relative to a fixed origin O

- the point P has position vector $3\mathbf{i} + 4\mathbf{k}$
- the point Q has position vector $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$
- the point R has position vector $-3\mathbf{i} - 18\mathbf{j} + a\mathbf{k}$

where a is a constant.

Given that P, Q and R lie on a straight line,

(a) find the value of a.

(3)

The line segment OP is extended to a point S so that $\overrightarrow{RS}$ is parallel to $\overrightarrow{OQ}$

(b) Find $\left|\overrightarrow{OS}\right|$

(3)

(Total for Question 13 is 6 marks)

3. Relative to a fixed origin O

- the point A has position vector $5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$
- the point B has position vector $2\mathbf{i} + 4\mathbf{j} + a\mathbf{k}$

where a is a positive integer.

(a) Show that $|\overrightarrow{OA}| = \sqrt{38}$

(1)

(b) Find the smallest value of a for which

$$|\overrightarrow{OB}| > |\overrightarrow{OA}|$$

(2)

Your Turn

3. Relative to a fixed origin O

- the point A has position vector $2\mathbf{i} + 3\mathbf{j} + a\mathbf{k}$
- the point B has position vector $5\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

where a is a positive integer.

(a) Show that $|\overrightarrow{OB}| = \sqrt{45}$

(1)

(b) Find the largest value of a for which

$$|\overrightarrow{OB}| > |\overrightarrow{OA}|$$

(2)

(Total for Question 3 is 3 marks)

12.4) Application to mechanics

Out of displacement, speed, acceleration, force, mass and time, all but mass and time are vectors. Clearly these can act in 3D space.

	Vector		Scalar	
Force	$\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} N$	$\Rightarrow$	$\sqrt{3^2 + 4^2 + (-1)^2}$ $= 5.10\ N$	
Acceleration	$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} ms^{-2}$	$\Rightarrow$	$1.41\ ms^{-2}$	
Displacement	$\begin{pmatrix} 12 \\ 3 \\ 4 \end{pmatrix} m$	$\Rightarrow$	$13\ m$	**Distance**
Velocity	$\begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} ms^{-1}$	$\Rightarrow$	$5\ m$	**Speed**

Worked Example

Convert these vectors to scalar form:

- A force of $\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} N$

- An acceleration of $\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} ms^{-2}$

- A displacement of $\begin{pmatrix} -6 \\ 8 \\ -24 \end{pmatrix} m$

- A velocity of $\begin{pmatrix} 8 \\ -6 \\ 0 \end{pmatrix} ms^{-1}$

Worked Example

A particle of mass 0.25 kg is acted on by three forces.

$$F_1 = (\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k})\ N$$
$$F_2 = (2\boldsymbol{i} - 4\boldsymbol{k})\ N$$
$$F_3 = (-5\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k})\ N$$

a) Find the resultant force R acting on the particle.

b) Find the acceleration of the particle, giving your answer in the form $(p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k})$ ms^{-2}.

c) Find the magnitude of the acceleration.

Given that the particle starts at rest,

d) Find the distance travelled by the particle in the first 3 seconds of its motion.

7.

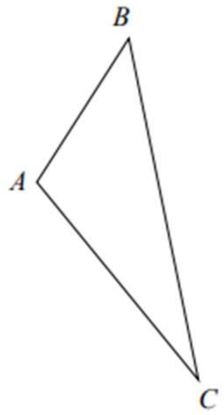


Figure 2

Figure 2 shows a sketch of a triangle ABC.

Given $\overrightarrow{AB} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\overrightarrow{BC} = \mathbf{i} - 9\mathbf{j} + 3\mathbf{k}$,

show that $\angle BAC = 105.9°$ to one decimal place.

(5)

Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

Attempts $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = 2\mathbf{i}+3\mathbf{j}+\mathbf{k}+\mathbf{i}-9\mathbf{j}+3\mathbf{k} = 3\mathbf{i}-6\mathbf{j}+4\mathbf{k}$	M1	3.1a
Attempts to find any one length using 3-d Pythagoras	M1	2.1
Finds all of $\lvert AB\rvert = \sqrt{14}$, $\lvert AC\rvert = \sqrt{61}$, $\lvert BC\rvert = \sqrt{91}$	A1ft	1.1b
$\cos BAC = \dfrac{14+61-91}{2\sqrt{14}\sqrt{61}}$	M1	2.1
angle $BAC = 105.9°$ *	A1*	1.1b
	(5)	

(5 marks)

Summary of Key Points

Summary of key points

1 The distance from the origin to the point (x, y, z) is $\sqrt{x^2 + y^2 + z^2}$

2 The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

3 The unit vectors along the x-, y- and z-axes are denoted by **i**, **j** and **k** respectively.

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad$ $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad$ $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Any 3D vector can be written in column form as $p\mathbf{i} + q\mathbf{j} + r\mathbf{k} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

4 If the vector $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ makes an angle θ_x with the positive x-axis then $\cos\theta_x = \frac{x}{|\mathbf{a}|}$ and similarly for the angles θ_y and θ_z.

5 If **a**, **b** and **c** are vectors in three dimensions which do not all lie in the same plane then you can compare their coefficients on both sides of an equation.